9. Values of Trigonometric Functions at Multiples and Submultiple of an Angles

Exercise 9.1

1. Question

Prove the following identities:

$$\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \tan x$$

Answer

To prove: $\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \tan x$

Proof:

Take LHS:

Let $I = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$

Identities used:

$\cos 2x = 1 - 2\sin^2 x$

$= 2\cos^2 x - 1$

Therefore,

$$= \sqrt{\frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x - 1)}}$$

$$= \sqrt{\frac{1-1+2\sin^2 x}{1+2\cos^2 x - 1}}$$

$$= \sqrt{\frac{2\sin^2 x}{2\cos^2 x}}$$

$$= \sqrt{\frac{\sin^2 x}{\cos^2 x}}$$

$$= \sqrt{\tan^2 x}$$

$$\left\{\because \frac{\sin x}{\cos x} = \tan x\right\}$$

$= \tan x$

$=$ RHS

Hence Proved

2. Question

Prove the following identities:

$$\frac{\sin 2x}{1-\cos 2x} = \cot x$$

Answer

To prove: $\frac{\sin 2x}{1-\cos 2x} = \cot x$

Proof:

Take LHS:

$$\frac{\sin 2x}{1-\cos 2x}$$

Identities used:

$\cos 2x = 1 - 2 \sin^2 x$

$\sin 2x = 2 \sin x \cos x$

Therefore,

$$= \frac{2 \sin x \cos x}{1-(1-2\sin^2 x)}$$

$$= \frac{2 \sin x \cos x}{1-1+2\sin^2 x}$$

$$= \frac{2 \sin x \cos x}{2 \sin^2 x}$$

$$= \frac{\cos x}{\sin x}$$

$$\left\{\because \frac{\cos x}{\sin x} = \cot x\right\}$$

= cot x

= RHS

Hence Proved

3. Question

Prove the following identities:

$$\frac{\sin 2x}{1+\cos 2x} = \tan x$$

Answer

To prove: $\frac{\sin 2x}{1+\cos 2x} = \tan x$

Proof:

Take LHS:

$$\frac{\sin 2x}{1+\cos 2x}$$

Identities used:

$\cos 2x = 2 \cos^2 x - 1$

$\sin 2x = 2 \sin x \cos x$

Therefore,

$$= \frac{2 \sin x \cos x}{1 + (2\cos^2 x - 1)}$$

$$= \frac{2 \sin x \cos x}{1 + 2\cos^2 x - 1}$$

$$= \frac{2 \sin x \cos x}{2 \cos^2 x}$$

$$= \frac{\sin x}{\cos x}$$

$$\left\{\because \frac{\sin x}{\cos x} = \tan x\right\}$$

= tan x

= RHS

Hence Proved

4. Question

Prove the following identities:

$$\sqrt{2 + \sqrt{2 + 2 \cos 4x}} = 2 \cos x, 0 < x < \frac{\pi}{4}$$

Answer

To prove: $\sqrt{2 + \sqrt{2 + 2 \cos 4x}} = 2 \cos x$

Proof:

Take LHS:

$$\sqrt{2 + \sqrt{2 + 2 \cos 4x}}$$

$$= \sqrt{2 + \sqrt{2 + 2(2\cos^2 2x - 1)}}$$

{∵ cos 2x = 2 $\cos^2$ x - 1 ⇒ cos 4x = 2 $\cos^2$ 2x -1}

$$= \sqrt{2 + \sqrt{2 + 4\cos^2 2x - 2}}$$

$$= \sqrt{2 + \sqrt{4 \cos^2 2x}}$$

$$= \sqrt{2 + 2 \cos 2x}$$

$$= \sqrt{2 + 2(2\cos^2 x - 1)}$$

{∵ cos 2x = 2 $\cos^2$ x - 1}

$$= \sqrt{2 + 4\cos^2 x - 2}$$

$$= \sqrt{4 \cos^2 x}$$

= 2 cos x

= RHS

Hence Proved

5. Question

Prove the following identities:

$$\frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}=\tan x$$

Answer

To prove: $\frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}=\tan x$

Proof:

Take LHS

$$\frac{1-\cos 2x+\sin 2x}{1+\cos 2x+\sin 2x}$$

Identities used:

$\cos 2x = 2\cos^2 x - 1$

$= 1 - 2\sin^2 x$

$\sin 2x = 2\sin x\cos x$

Therefore,

$$=\frac{1-(1-2\sin^2 x)+2\sin x\cos x}{1+(2\cos^2 x-1)+2\sin x\cos x}$$

$$=\frac{1-1+2\sin^2 x+2\sin x\cos x}{1+2\cos^2 x-1+2\sin x\cos x}$$

$$=\frac{2\sin^2 x+2\sin x\cos x}{2\cos^2 x+2\sin x\cos x}$$

$$=\frac{2\sin x(\sin x+\cos x)}{2\cos x(\cos x+\sin x)}$$

$$=\frac{\sin x}{\cos x}$$

$= \tan x$

$$\left\{\because \frac{\sin x}{\cos x}=\tan x\right\}$$

= RHS

Hence Proved

6. Question

Prove the following identities:

$$\frac{\sin x+\sin 2x}{1+\cos x+\cos 2x}=\tan x$$

Answer

To prove: $\frac{\sin x+\sin 2x}{1+\cos x+\cos 2x}=\tan x$

Proof:

Take LHS:

$$\frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x}$$

Identities used:

$\cos 2x = \cos^2 x - \sin^2 x$

$\sin 2x = 2 \sin x \cos x$

Therefore,

$$= \frac{\sin x + 2 \sin x \cos x}{1 + \cos x + (2\cos^2 x - 1)}$$

$$= \frac{\sin x + 2 \sin x \cos x}{1 + \cos x + 2\cos^2 x - 1}$$

$$= \frac{\sin x + 2 \sin x \cos x}{\cos x + 2\cos^2 x}$$

$$= \frac{\sin x (1 + 2\cos x)}{\cos x (1 + 2\cos x)}$$

$$= \frac{\sin x}{\cos x}$$

= tan x

$$\left\{\because \frac{\sin x}{\cos x} = \tan x\right\}$$

= RHS

Hence Proved

7. Question

Prove the following identities:

$$\frac{\cos 2x}{1 + \sin 2x} = \tan\left(\frac{\pi}{4} - x\right)$$

Answer

To prove: $\frac{\cos 2x}{1 + \sin 2x} = \tan\left(\frac{\pi}{4} - x\right)$

Proof:

Take LHS:

$$\frac{\cos 2x}{1 + \sin 2x}$$

Identities used:

$\cos 2x = \cos^2 x - \sin^2 x$

$\sin 2x = 2 \sin x \cos x$

Therefore,

$$= \frac{\cos^2 x - \sin^2 x}{1 + 2 \sin x \cos x}$$

$$=\frac{(\cos x-\sin x)(\cos x+\sin x)}{\sin^2 x+\cos^2 x+2\sin x\cos x}$$

{∵ $a^2 - b^2 = (a - b)(a + b)$ & $\sin^2 x + \cos^2 x = 1$}

$$=\frac{(\cos x-\sin x)(\cos x+\sin x)}{(\sin x+\cos x)^2}$$

{∵ $a^2 + b^2 + 2ab = (a + b)^2$}

$$=\frac{(\cos x-\sin x)(\cos x+\sin x)}{(\sin x+\cos x)(\sin x+\cos x)}$$

$$=\frac{(\cos x-\sin x)}{(\sin x+\cos x)}$$

Multiplying numerator and denominator by $\frac{1}{\sqrt{2}}$:

$$=\frac{\frac{1}{\sqrt{2}}(\cos x-\sin x)}{\frac{1}{\sqrt{2}}(\sin x+\cos x)}$$

$$=\frac{\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)}{\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)}$$

$$=\frac{\left(\sin\frac{\pi}{4}\cos x-\cos\frac{\pi}{4}\sin x\right)}{\left(\sin\frac{\pi}{4}\sin x+\cos\frac{\pi}{4}\cos x\right)}$$

$$\left\{\because \sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}\right\}$$

$$=\frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos\left(\frac{\pi}{4}-x\right)}$$

{∵ sin (A – B) = sin A cos B – sin B cos A

cos (A – B) = cos A cos B + sin A sin B}

$$=\tan\left(\frac{\pi}{4}-x\right)$$

$$\left\{\because \frac{\sin x}{\cos x}=\tan x\right\}$$

= RHS

Hence Proved

8. Question

Prove the following identities:

$$\frac{\cos x}{1-\sin x}=\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

Answer

To prove: $\frac{\cos x}{1-\sin x}=\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

Proof:

Take LHS:

$$\frac{\cos x}{1-\sin x}$$

Identities used:

$\cos 2x = \cos^2 x - \sin^2 x$

$$\Rightarrow \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$\sin 2x = 2 \sin x \cos x$

$$\Rightarrow \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

Therefore,

$$= \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$\{\because a^2 - b^2 = (a - b)(a + b)$ & $\sin^2 x + \cos^2 x = 1\}$

$$= \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}$$

$\{\because a^2 + b^2 + 2ab = (a + b)^2\}$

$$= \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)}$$

$$= \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)}$$

$$= \frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)}$$

Multiplying numerator and denominator by $\frac{1}{\sqrt{2}}$:

$$= \frac{\frac{1}{\sqrt{2}}\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\frac{1}{\sqrt{2}}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)}$$

$$= \frac{\left(\frac{1}{\sqrt{2}}\cos\frac{x}{2} + \frac{1}{\sqrt{2}}\sin\frac{x}{2}\right)}{\left(\frac{1}{\sqrt{2}}\sin\frac{x}{2} - \frac{1}{\sqrt{2}}\cos\frac{x}{2}\right)}$$

$$=\frac{\left(\sin\frac{\pi}{4}\cos\frac{x}{2}+\cos\frac{\pi}{4}\sin\frac{x}{2}\right)}{\left(\sin\frac{\pi}{4}\sin\frac{x}{2}-\cos\frac{\pi}{4}\cos\frac{x}{2}\right)}$$

$$\left\{\because \sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}\right\}$$

$$=\frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos\left(\frac{\pi}{4}-x\right)}$$

{∵ sin (A – B) = sin A cos B – sin B cos A

cos (A – B) = cos A cos B + sin A sin B}

$$=\tan\left(\frac{\pi}{4}-x\right)$$

$$\left\{\because \frac{\sin x}{\cos x}=\tan x\right\}$$

= RHS

Hence Proved

9. Question

Prove the following identities:

$$\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}+\cos^2\frac{5\pi}{8}+\cos^2\frac{7\pi}{8}=2$$

Answer

To prove: $\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}+\cos^2\frac{5\pi}{8}+\cos^2\frac{7\pi}{8}=2$

Proof:

Take LHS:

$$\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}+\cos^2\frac{5\pi}{8}+\cos^2\frac{7\pi}{8}$$

Identities used:

cos 2x = 2 $\cos^2$ x – 1

⇒ 2 $\cos^2$ x = 1 + cos 2x

$$\Rightarrow \cos^2 x=\frac{1+\cos 2x}{2}$$

Therefore,

$$=\frac{1+\cos\frac{2\pi}{8}}{2}+\frac{1+\cos\frac{6\pi}{8}}{2}+\frac{1+\cos\frac{10\pi}{8}}{2}+\frac{1+\cos\frac{14\pi}{8}}{2}$$

$$=\frac{1+\cos\frac{2\pi}{8}}{2}+\frac{1+\cos\left(\pi-\frac{2\pi}{8}\right)}{2}+\frac{1+\cos\left(\pi+\frac{2\pi}{8}\right)}{2}+\frac{1+\cos\left(2\pi-\frac{2\pi}{8}\right)}{2}$$

$$\left\{\because \pi-\frac{2\pi}{8}=\frac{6\pi}{8};\pi+\frac{2\pi}{8}=\frac{10\pi}{8};2\pi-\frac{2\pi}{8}=\frac{14\pi}{8}\right\}$$

$$= \frac{1+\cos\frac{2\pi}{8}}{2} + \frac{1-\cos\frac{2\pi}{8}}{2} + \frac{1-\cos\frac{2\pi}{8}}{2} + \frac{1+\cos\frac{2\pi}{8}}{2}$$

{∵ cos (π – θ) = - cos θ, cos (π + θ) = - cos θ & cos(2π – θ) = cos θ}

$$= 2 \times \frac{1+\cos\frac{2\pi}{8}}{2} + 2 \times \frac{1-\cos\frac{2\pi}{8}}{2}$$

$$= 1 + \cos\frac{2\pi}{8} + 1 - \cos\frac{2\pi}{8}$$

= 2

= RHS

Hence Proved

10. Question

Prove the following identities:

$$\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{5\pi}{8} + \sin^2\frac{7\pi}{8} = 2$$

Answer

To prove: $\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{5\pi}{8} + \sin^2\frac{7\pi}{8} = 2$

Proof:

Take LHS:

$$\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{5\pi}{8} + \sin^2\frac{7\pi}{8}$$

Identities used:

cos 2x = 1 – 2 $\sin^2$ x

⇒ 2 $\sin^2$ x = 1 – cos 2x

$$\Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$$

Therefore,

$$= \frac{1-\cos\frac{2\pi}{8}}{2} + \frac{1-\cos\frac{6\pi}{8}}{2} + \frac{1-\cos\frac{10\pi}{8}}{2} + \frac{1-\cos\frac{14\pi}{8}}{2}$$

$$= \frac{1-\cos\frac{2\pi}{8}}{2} + \frac{1-\cos\left(\pi-\frac{2\pi}{8}\right)}{2} + \frac{1-\cos\left(\pi+\frac{2\pi}{8}\right)}{2} + \frac{1-\cos\left(2\pi-\frac{2\pi}{8}\right)}{2}$$

$$\left\{\because \pi - \frac{2\pi}{8} = \frac{6\pi}{8}; \pi + \frac{2\pi}{8} = \frac{10\pi}{8}; 2\pi - \frac{2\pi}{8} = \frac{14\pi}{8}\right\}$$

$$= \frac{1-\cos\frac{2\pi}{8}}{2} + \frac{1-\left(-\cos\frac{2\pi}{8}\right)}{2} + \frac{1-\left(-\cos\frac{2\pi}{8}\right)}{2} + \frac{1-\cos\frac{2\pi}{8}}{2}$$

{∵ cos (π – θ) = - cos θ,

cos (π + θ) = - cos θ &

cos(2π – θ) = cos θ}

$= \frac{1-\cos\frac{2\pi}{8}}{2} + \frac{1+\cos\frac{2\pi}{8}}{2} + \frac{1+\cos\frac{2\pi}{8}}{2} + \frac{1-\cos\frac{2\pi}{8}}{2}$

$= 2 \times \frac{1-\cos\frac{2\pi}{8}}{2} + 2 \times \frac{1+\cos\frac{2\pi}{8}}{2}$

$= 1 - \cos\frac{2\pi}{8} + 1 + \cos\frac{2\pi}{8}$

= 2

= RHS

Hence Proved

11. Question

Prove the following identities:

$(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \cos^2\left(\frac{\alpha-\beta}{2}\right)$

Answer

To prove: $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\frac{\alpha-\beta}{2}$

Proof:

Take LHS:

$(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$

$= \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta$

$= 2 + 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$

$= 2(1 + \cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= 2(1 + \cos(\alpha-\beta))$

$\{\because \cos(A - B) = \cos A \cos B + \sin A \sin B\}$

$= 2\left(1 + 2\cos^2\frac{\alpha-\beta}{2} - 1\right)$

$\{\because \cos 2x = 2\cos^2 x - 1\}$

$= 2\left(2\cos^2\frac{\alpha-\beta}{2}\right)$

$= 4\cos^2\frac{\alpha-\beta}{2}$

= RHS

Hence Proved

12. Question

Prove the following identities:

$\sin^2\left(\frac{\pi}{8}+\frac{x}{2}\right) - \sin^2\left(\frac{\pi}{8}-\frac{x}{2}\right) = \frac{1}{\sqrt{2}}\sin x$

Answer

To prove: $\sin^2\left(\frac{\pi}{8}+\frac{x}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{x}{2}\right)=\frac{1}{\sqrt{2}}\sin x$

Proof:

Take LHS:

$\sin^2\left(\frac{\pi}{8}+\frac{x}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{x}{2}\right)$

Identities used:

$\sin^2 A - \sin^2 B = \sin(A + B)\sin(A - B)$

Therefore,

$=\sin\left(\frac{\pi}{8}+\frac{x}{2}+\frac{\pi}{8}-\frac{x}{2}\right)\sin\left(\frac{\pi}{8}+\frac{x}{2}-\left(\frac{\pi}{8}-\frac{x}{2}\right)\right)$

$=\sin\left(\frac{\pi}{8}+\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}+\frac{x}{2}-\frac{\pi}{8}+\frac{x}{2}\right)$

$=\sin\frac{\pi}{4}\sin x$

$=\frac{1}{\sqrt{2}}\sin x$

= RHS

Hence Proved

13. Question

Prove the following identities:

$1 + \cos^2 2x = 2(\cos^4 x + \sin^4 x)$

Answer

To prove: $1 + \cos^2 2x = 2(\cos^4 x + \sin^4 x)$

Proof:

Take LHS:

$1+\cos^2 2x$

$=[(\cos^2 x+\sin^2 x)]^2 + [(\cos^2 x-\sin^2 x)]^2$

{$\because$ $\cos 2x = \cos^2 x - \sin^2 x$ & $\cos^2 x + \sin^2 x = 1$}

$=(\cos^4 x+\sin^4 x+2\cos^2 x\sin^2 x)+(\cos^4 x+\sin^4 x-2\cos^2 x\sin^2 x)$

$=\cos^4 x+\sin^4 x+\cos^4 x+\sin^4 x$

$=2\cos^4 x+2\sin^4 x$

$=2(\cos^4 x+\sin^4 x)$

= RHS

14. Question

Prove the following identities:

$\cos^3 2x + 3\cos 2x = 4(\cos^6 x - \sin^6 x)$

Answer

To prove: $\cos^3 2x + 3\cos 2x = 4(\cos^6 x - \sin^6 x)$

Proof:

Take RHS:

$4(\cos^6 x - \sin^6 x)$

$= 4((\cos^2 x)^3 - (\sin^2 x)^3)$

$= 4(\cos^2 x - \sin^2 x)(\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x)$

$= 4(\cos^2 x - \sin^2 x)(\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x)$

{∵ $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$}

$= 4\cos 2x(\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x + \cos^2 x \sin^2 x - \cos^2 x \sin^2 x)$

{∵ $\cos 2x = \cos^2 x - \sin^2 x$}

$= 4\cos 2x(\cos^4 x + \sin^4 x + 2\cos^2 x \sin^2 x - \cos^2 x \sin^2 x)$

$= 4\cos 2x\{(\cos^2 x)^2 + (\sin^2 x)^2 + 2\cos^2 x \sin^2 x - \cos^2 x \sin^2 x)$

{∵ $a^2 + b^2 + 2ab = (a + b)^2$}

$= 4\cos 2x\{(\cos^2 x + \sin^2 x)^2 - \cos^2 x \sin^2 x\}$

{∵ $\cos^2 x + \sin^2 x = 1$}

$= 4\cos 2x\{(1)^2 - \frac{1}{4}(4\cos^2 x \sin^2 x)\}$

$= 4\cos 2x\{(1)^2 - \frac{1}{4}(2\cos x \sin x)^2\}$

{∵ $\sin 2x = 2\sin x\cos x$}$= 4\cos 2x\left\{(1)^2 - \frac{1}{4}(\sin 2x)^2\right\}$

$= 4\cos 2x\left(1 - \frac{1}{4}\sin^2 2x\right)$

{∵ $\sin^2 x = 1 - \cos^2 x$}

$= 4\cos 2x\left(1 - \frac{1}{4}(1 - \cos^2 2x)\right)$

$= 4\cos 2x\left(1 - \frac{1}{4} + \frac{1}{4}\cos^2 2x\right)$

$= 4\cos 2x\left(\frac{3}{4} + \frac{1}{4}\cos^2 2x\right)$

$= 4\left(\frac{3}{4}\cos 2x + \frac{1}{4}\cos^3 2x\right)$

$= 3\cos 2x + \cos^3 2x$

= LHS

Hence Proved

15. Question

Prove the following identities:

$(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$

Answer

To prove: (sin 3x + sin x)sin x + (cos 3x – cos x)cos x= 0

Proof:

Take LHS:

(sin 3x + sin x)sin x + (cos 3x – cos x)cos x

= (sin 3x)(sin x) + $\sin^2$ x + (cos 3x)(cos x) – $\cos^2$ x

= [(sin 3x)(sin x) + (cos 3x)(cos x)] + ($\sin^2$ x – $\cos^2$ x)

= [(sin 3x)(sin x) + (cos 3x)(cos x)] – ($\cos^2$ x – $\sin^2$ x)

= cos(3x – x) – cos 2x

{∵ cos 2x = $\cos^2$ x – $\sin^2$ x &

cos A cos B + sin A sin B = cos(A – B)}

= cos 2x – cos 2x

= 0

= RHS

Hence Proved

16. Question

Prove the following identities:

$$\cos^2\left(\frac{\pi}{4}-x\right)-\sin^2\left(\frac{\pi}{4}-x\right)=\sin 2x$$

Answer

To prove: $\cos^2\left(\frac{\pi}{4}-x\right)-\sin^2\left(\frac{\pi}{4}-x\right)=\sin 2x$

Proof:

Take LHS:

$\cos^2\left(\frac{\pi}{4}-x\right)-\sin^2\left(\frac{\pi}{4}-x\right)$

Identities used:

$\cos^2$ A – $\sin^2$ A = cos 2A

Therefore,

$= \cos 2\left(\frac{\pi}{4}-x\right)$

$= \cos\left(\frac{\pi}{2}-2x\right)$

$= \sin 2x$

$\left\{\because \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta\right\}$

= RHS

Hence Proved

17. Question

Prove the following identities:

$\cos 4x = 1 - 8\cos^2 x + 8\cos^4 x$

Answer

To prove: $\cos 4x = 1 - 8\cos^2 x + 8\cos^4 x$

Proof:

Take LHS:

$\cos 4x$

Identities used:

$\cos 2x = = 2\cos^2 x - 1$

Therefore,

$= 2\cos^2 2x - 1$

$= 2(2\cos^2 2x - 1)^2 - 1$

$= 2\{(2\cos^2 2x\}^2 + 1^2 - 2\times 2\cos^2 x\} - 1$

$= 2(4\cos^4 2x + 1 - 4\cos^2 x) - 1$

$= 8\cos^4 2x + 2 - 8\cos^2 x - 1$

$= 8\cos^4 2x + 1 - 8\cos^2 x$

= RHS

Hence Proved

18. Question

Prove the following identities:

$\sin 4x = 4\sin x\cos^3 x - 4\cos x\sin^3 x$

Answer

To prove: $\sin 4x = 4\sin x\cos^3 x - 4\cos x\sin^3 x$

Proof:

Take LHS:

$\sin 4x$

Identities used:

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

Therefore,

$= 2\sin 2x\cos 2x$

$= 2(2\sin x\cos x)(\cos^2 x - \sin^2 x)$

$= 4\sin x\cos x(\cos^2 x - \sin^2 x)$

$= 4\sin x\cos^3 x - 4\sin^3 x\cos x$

= RHS

Hence Proved

19. Question

Prove the following identities:

$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13$

Answer

To prove: $3(\sin x - \cos x)^4 + 6 (\sin x + \cos x)^2 + 4 (\sin^6 x + \cos^6 x) = 13$

Proof:

Take LHS:

$3(\sin x - \cos x)^4 + 6 (\sin x + \cos x)^2 + 4 (\sin^6 x + \cos^6 x)$

Identities used:

$(a + b)^2 = a^2 + b^2 + 2ab$

$(a - b)^2 = a^2 + b^2 - 2ab$

$a^3 + b^3 = (a + b) (a^2 + b^2 - ab)$

Therefore,

$= 3\{(\sin x - \cos x)^2\}^2 + 6 \{(\sin x)^2 + (\cos x)^2 + 2 \sin x \cos x) + 4 \{(\sin^2 x)^3 + (\cos^2 x)^3\}$

$= 3\{(\sin x)^2 + (\cos x)^2 - 2 \sin x \cos x)\}^2 + 6 (\sin^2 x + \cos^2 x + 2 \sin x \cos x) + 4\{(\sin^2 x + \cos^2 x) (\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)\}$

$= 3(1 - 2 \sin x \cos x)^2 + 6 (1 + 2 \sin x \cos x) + 4\{(1) (\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)\}$

$\{\because \sin^2 x + \cos^2 x = 1\}$

$= 3\{1^2 + (2 \sin x \cos x)^2 - 4 \sin x \cos x\} + 6 (1 + 2 \sin x \cos x) + 4\{(\sin^2 x)^2 + (\cos^2 x)^2 + 2 \sin^2 x \cos^2 x - 3 \sin^2 x \cos^2 x)\}$

$= 3\{1 + 4 \sin^2 x \cos^2 x - 4 \sin x \cos x\} + 6 (1 + 2 \sin x \cos x) + 4\{(\sin^2 x + \cos^2 x)^2 - 3 \sin^2 x \cos^2 x)\}$

$= 3 + 12 \sin^2 x \cos^2 x - 12 \sin x \cos x + 6 + 12 \sin x \cos x + 4\{(1)^2 - 3 \sin^2 x \cos^2 x)\}$

$= 9 + 12 \sin^2 x \cos^2 x + 4(1 - 3 \sin^2 x \cos^2 x)$

$= 9 + 12 \sin^2 x \cos^2 x + 4 - 12 \sin^2 x \cos^2 x$

$= 13$

= RHS

Hence Proved

20. Question

Prove the following identities:

$2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$

Answer

To prove: $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$

Proof:

Take LHS:

$2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1$

Identities used:

$(a + b)^2 = a^2 + b^2 + 2ab$

$a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

Therefore,

$= 2\{(\sin^2 x)^3 + (\cos^2 x)^3\} - 3\{(\sin^2 x)^2 + (\cos^2 x)^2\} + 1$

$= 2\{(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x\} - 3\{(\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x \cos^2 x - 2\sin^2 x \cos^2 x\} + 1$

$= 2\{(1)(\sin^4 x + \cos^4 x + 2 \sin^2 x \cos^2 x - 3 \sin^2 x \cos^2 x\} - 3\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x\} + 1$

$\{\because \sin^2 x + \cos^2 x = 1\}$

$= 2\{(\sin^2 x + \cos^2 x)^2 - 3 \sin^2 x \cos^2 x\} - 3\{(1)^2 - 2\sin^2 x \cos^2 x\} + 1$

$= 2\{(1)^2 - 3 \sin^2 x \cos^2 x\} - 3(1 - 2\sin^2 x \cos^2 x) + 1$

$= 2(1 - 3 \sin^2 x \cos^2 x) - 3 + 6 \sin^2 x \cos^2 x + 1$

$= 2 - 6 \sin^2 x \cos^2 x - 2 + 6 \sin^2 x \cos^2 x$

$= 0$

= RHS

Hence Proved

21. Question

Prove the following identities:

$\cos^6 x - \sin^6 x = \cos 2x \left(1 - \frac{1}{4}\sin^2 2x\right)$

Answer

To prove: $\cos^6 x - \sin^6 x = \cos 2x\left(1 - \frac{1}{4}\sin^2 2x\right)$

Proof:

Take LHS:

$\cos^6 x - \sin^6 x$

Identities used:

$(a + b)^2 = a^2 + b^2 + 2ab$

$a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$

Therefore,

$= (\cos^2 x)^3 - (\sin^2 x)^3$

$= (\cos^2 x - \sin^2 x)(\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x)$

$\{\because \cos 2x = \cos^2 x - \sin^2 x\}$

$= \cos 2x((\cos^2 x)^2 + (\sin^2 x)^2 + 2\cos^2 x \sin^2 x - \cos^2 x \sin^2 x)$

$= \cos 2x\left((\cos^2 x + \sin^2 x)^2 - \frac{1}{4} \times 4\cos^2 x \sin^2 x\right)$

$\{\because \sin^2 x + \cos^2 x = 1\}$

$= \cos 2x\left((1)^2 - \frac{1}{4} \times (2\cos x \sin x)^2\right)$

{∵ sin 2x = 2 sin x cos x}

$= \cos 2x\left(1 - \frac{1}{4} \times (\sin 2x)^2\right)$

$= \cos 2x\left(1 - \frac{1}{4}\sin^2 2x\right)$

= RHS

Hence Proved

22. Question

Prove the following identities:

$$\tan\left(\frac{\pi}{4} + x\right) + \tan\left(\frac{\pi}{4} - x\right) = 2 \sec 2x$$

Answer

To prove: $\tan\left(\frac{\pi}{4} + x\right) + \tan\left(\frac{\pi}{4} - x\right) = 2\sec 2x$

Proof:

Take LHS:

$\tan\left(\frac{\pi}{4} + x\right) + \tan\left(\frac{\pi}{4} - x\right)$

Identities used:

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Therefore,

$= \frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x} + \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}$

$\left\{\because \tan\frac{\pi}{4} = 1\right\}$

$= \frac{1 + \tan x}{1 - \tan x} + \frac{1 - \tan x}{1 + \tan x}$

$= \frac{(1 + \tan x)^2 + (1 - \tan x)^2}{(1 - \tan x)(1 + \tan x)}$

{∵ $(a - b)(a + b) = a^2 - b^2$;

$(a + b)^2 = a^2 + b^2 + 2ab$ &

$(a - b)^2 = a^2 + b^2 - 2ab$}

$= \frac{1^2 + \tan^2 x + 2\tan x + 1^2 + \tan^2 x - 2\tan x}{1^2 - \tan^2 x}$

$= \frac{1 + \tan^2 x + 1 + \tan^2 x}{1 - \tan^2 x}$

$$= \frac{2(1 + \tan^2 x)}{1 - \tan^2 x}$$

$$\left\{\because \tan x = \frac{\sin x}{\cos x}\right\}$$

$$= \frac{2\left(1 + \left(\frac{\sin x}{\cos x}\right)^2\right)}{1 - \left(\frac{\sin x}{\cos x}\right)^2}$$

$$= \frac{2\left(1 + \frac{\sin^2 x}{\cos^2 x}\right)}{1 - \frac{\sin^2 x}{\cos^2 x}}$$

$$= \frac{2\left(\frac{\cos^2 x + \sin^2 x}{\cos^2 x}\right)}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}$$

{∵ $\cos^2 x + \sin^2 x = 1$ & $\cos 2x = \cos^2 x - \sin^2 x$}

$$= \frac{2\left(\frac{1}{\cos^2 x}\right)}{\frac{\cos 2x}{\cos^2 x}}$$

$$= \frac{2}{\cos 2x}$$

= 2 sec 2x

$$\left\{\because \frac{1}{\cos 2x} = \sec 2x\right\}$$

= RHS

Hence Proved

23. Question

Prove the following identities:

$\cot^2 x - \tan^2 x = 4 \cot 2x \operatorname{cosec} 2x$

Answer

To prove: $\cot^2 x - \tan^2 x = 4 \cot 2x \operatorname{cosec} 2x$

Proof:

Take LHS:

$\cot^2 x - \tan^2 x$

Identities used:

$a^2 - b^2 = (a - b)(a + b)$

Therefore,

= (cot x – tan x)(cot x + tan x)

$$\left\{\because \tan x = \frac{1}{\cot x}\right\}$$

$$= \left(\cot x - \frac{1}{\cot x}\right)\left(\cot x + \frac{1}{\cot x}\right)$$

$= \left(\frac{\cot^2 x - 1}{\cot x}\right)\left(\frac{\cot^2 x + 1}{\cot x}\right)$

$= 2\left(\frac{\cot^2 x - 1}{2\cot x}\right)\left(\frac{\cot^2 x + 1}{\cot x}\right)$

{∵ $\cot^2 x + 1 = \text{cosec}^2 x$}

$= 2\left(\frac{\cot^2 x - 1}{2\cot x}\right)\left(\frac{\text{cosec}^2 x}{\cot x}\right)$

$= 2(\cot 2x)\left(\frac{\frac{1}{\sin^2 x}}{\frac{\cos x}{\sin x}}\right)$

$\left\{\begin{array}{c} \because \cot 2x = \frac{\cot^2 x - 1}{2\cot x}; \\ \text{cosec}\, x = \frac{1}{\sin x}; \\ \cot x = \frac{\cos x}{\sin x} \end{array}\right\}$

$= 2(\cot 2x)\left(\frac{1}{\sin x \cos x}\right)$

$= 2(\cot 2x)\left(\frac{2}{2\cos x \sin x}\right)$

$= \frac{4\cot 2x}{\sin 2x}$

{∵ $\sin 2x = 2 \sin x \cos x$}

$= 4 \cot 2x \text{ cosec } 2x$

$\left\{\because \text{cosec}\, x = \frac{1}{\sin x}\right\}$

= RHS

Hence Proved

24. Question

Prove the following identities:

$\cos 4x - \cos 4\alpha = 8(\cos x - \cos \alpha)(\cos x + \cos \alpha)(\cos x - \sin \alpha)(\cos x + \sin \alpha)$

Answer

To prove: $\cos 4x - \cos 4\alpha = 8(\cos x - \cos \alpha)(\cos x + \cos \alpha)(\cos x - \sin \alpha)(\cos x + \sin \alpha)$

Proof:

Take LHS:

$\text{Cos } 4x - \cos 4\alpha$

{∵ $\cos 2\theta = 2 \cos^2 \theta - 1$}

$= 2 \cos^2 2x - 1 - (2 \cos^2 2\alpha - 1)$

$= 2 \cos^2 2x - 1 - 2 \cos^2 2\alpha + 1$

$= 2 \cos^2 2x - 2 \cos^2 2\alpha$

$= 2(\cos^2 2x - \cos^2 2\alpha)$

$\{\because (a - b)(a + b) = a^2 - b^2\}$

$= 2(\cos 2x - \cos 2\alpha)(\cos 2x + \cos 2\alpha)$

$\{\because \cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta\}$

$= 2\{2\cos^2 x - 1 - (2\cos^2\alpha - 1)\}(2\cos^2 x - 1 + 1 - 2\sin^2\alpha)$

$= 2\{2\cos^2 x - 1 - 2\cos^2\alpha + 1\}(2\cos^2 x - 2\sin^2\alpha)$

$= 2 \times 2\{2\cos^2 x - 2\cos^2\alpha\}(\cos^2 x - \sin^2\alpha)$

$= 4 \times 2\{\cos^2 x - \cos^2\alpha\}(\cos^2 x - \sin^2\alpha)$

$= 8(\cos x - \cos\alpha)(\cos x + \cos\alpha)(\cos x - \sin\alpha)(\cos x + \sin\alpha)$

= RHS

Hence Proved

25. Question

Prove the following identities:

$\sin 3x + \sin 2x - \sin x = 4\sin x\cos\frac{x}{2}\cos\frac{3x}{2}$

Answer

To prove: $\sin 3x + \sin 2x - \sin x = 4\sin x\cos\frac{x}{2}\cos\frac{3x}{2}$

Proof:

Take LHS:

$\sin 3x + \sin 2x - \sin x$

Identities used:

$\sin 2x = 2\sin x\cos x$

$\sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}$

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

Therefore,

$= 2\sin\frac{3x}{2}\cos\frac{3x}{2} + 2\sin\frac{2x-x}{2}\cos\frac{2x+x}{2}$

$= 2\sin\frac{3x}{2}\cos\frac{3x}{2} + 2\sin\frac{x}{2}\cos\frac{3x}{2}$

$= 2\cos\frac{3x}{2}\left(\sin\frac{3x}{2} + \sin\frac{x}{2}\right)$

$= 2\cos\frac{3x}{2}\left(2\sin\frac{\frac{3x}{2}+\frac{x}{2}}{2}\cos\frac{\frac{3x}{2}-\frac{x}{2}}{2}\right)$

$= 2\cos\frac{3x}{2}\left(2\sin\frac{\frac{4x}{2}}{2}\cos\frac{\frac{2x}{2}}{2}\right)$

$= 2\cos\frac{3x}{2}\left(2\sin\frac{2x}{2}\cos\frac{x}{2}\right)$

$= 4\sin x\cos\frac{x}{2}\cos\frac{3x}{2}$

= RHS

Hence Proved

26. Question

Prove that: $\tan 82\frac{1}{2}^{\circ} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1) = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

Answer

To prove: $\tan 82\frac{1}{2}^{\circ} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1) = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

Proof:

Identities used:

$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$

Therefore,

tan 15° = tan (45° - 30°)

$\Rightarrow \tan 15° = \frac{\tan 45° - \tan 30°}{1+\tan 30°\tan 45°}$

$\Rightarrow \tan 15° = \frac{1-\frac{1}{\sqrt{3}}}{1+(1)\left(\frac{1}{\sqrt{3}}\right)}$

$\left\{\because \tan 45° = 1 \;\&\; \tan 30° = \frac{1}{\sqrt{3}}\right\}$

$\Rightarrow \tan 15° = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$

$\Rightarrow \tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

On rationalising:

$\Rightarrow \tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}$

$\Rightarrow \tan 15° = \frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2-1}$

$\{\because (a-b)(a+b) = a^2 - b^2\}$

$\Rightarrow \tan 15° = \frac{3+1-2\sqrt{3}}{3-1}$

$\Rightarrow \tan 15° = \frac{4 - 2\sqrt{3}}{2}$

$\Rightarrow \tan 15° = \frac{2(2 - \sqrt{3})}{2}$

$\Rightarrow \tan 15° = 2 - \sqrt{3}$

$\Rightarrow \cot 15° = \frac{1}{2 - \sqrt{3}}$

$\left\{\because \cot x = \frac{1}{\tan x}\right\}$

On rationalising

$\Rightarrow \cot 15° = \frac{1}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$

$\Rightarrow \cot 15° = \frac{2 + \sqrt{3}}{(2)^2 - (\sqrt{3})^2}$

$\{\because (a - b)(a + b) = a^2 - b^2\}$

$\Rightarrow \cot 15° = \frac{2 + \sqrt{3}}{4 - 3}$

$\Rightarrow \cot 15° = 2 + \sqrt{3}$

Let 2θ = 15°

$\Rightarrow \cot 2\theta = 2 + \sqrt{3}$

We know,

$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$

$\Rightarrow \frac{\cot^2\theta - 1}{2\cot\theta} = 2 + \sqrt{3}$

$\Rightarrow \cot^2\theta - 1 = 2(2 + \sqrt{3})\cot\theta$

$\Rightarrow \cot^2\theta - 2(2 + \sqrt{3})\cot\theta - 1 = 0$

Formula used:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for $ax^2 + bx + c = 0$

$\Rightarrow \cot\theta = \frac{-[-2(2 + \sqrt{3})] \pm \sqrt{[-2(2 + \sqrt{3})]^2 - 4(1)(-1)}}{2(1)}$

$\Rightarrow \cot\theta = \frac{2(2 + \sqrt{3}) \pm \sqrt{4(4 + 3 + 4\sqrt{3}) + 4}}{2}$

$\{\because (a + b)^2 = a^2 + b^2 + 2ab\}$

$\Rightarrow \cot\theta = \frac{2(2 + \sqrt{3}) \pm 2\sqrt{7 + 4\sqrt{3} + 1}}{2}$

$$\Rightarrow \cot\theta = \left(2+\sqrt{3}\right) \pm \sqrt{8+4\sqrt{3}}$$

cot θ < 0 as θ is in 1st quadrant.

So,

$$\cot\theta = \left(2+\sqrt{3}\right) + \sqrt{8+4\sqrt{3}}$$

$$\Rightarrow \cot\theta = \left(2+\sqrt{3}\right) + \sqrt{\left(\sqrt{6}\right)^2 + \left(\sqrt{2}\right)^2 + 2\cdot\left(\sqrt{6}\right)\left(\sqrt{2}\right)}$$

{∵ $(a + b)^2 = a^2 + b^2 + 2ab$}

$$\Rightarrow \cot\theta = \left(2+\sqrt{3}\right) + \sqrt{\left(\sqrt{6}+\sqrt{2}\right)^2}$$

$$\Rightarrow \cot\theta = \left(2+\sqrt{3}\right) + \left(\sqrt{6}+\sqrt{2}\right)$$

$$\text{As, } 2\theta = 15° \Rightarrow \theta = \frac{15°}{2} = 7\frac{1}{2}°$$

$$\Rightarrow \cot 7\frac{1}{2}° = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$$

$$\{\because 4 = \sqrt{2}\}$$

$$\Rightarrow \tan\left(90° - 7\frac{1}{2}°\right) = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$$

{∵ cot θ = tan(90° - θ)}

$$\Rightarrow \tan 82\frac{1}{2}° = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$$

Hence Proved

27. Question

Prove that: $\cot\frac{\pi}{8} = \sqrt{2}+1$

Answer

To prove: $\cot\frac{\pi}{8} = \sqrt{2}+1$

Proof:

Take LHS:

Let 2θ = 45°

We know,

$$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$$

$$\Rightarrow \cot 45° = \frac{\cot^2\theta - 1}{2\cot\theta}$$

{∵ cot 45° = 1}

$$\Rightarrow 1 = \frac{\cot^2\theta - 1}{2\cot\theta}$$

$\Rightarrow 2\cot\theta = \cot^2\theta - 1$

$\Rightarrow \cot^2\theta - 2\cot\theta - 1 = 0$

Formula used:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for $ax^2 + bx + c = 0$

$\Rightarrow \cot\theta = \frac{-[-2] \pm \sqrt{[-2]^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$

$\Rightarrow \cot\theta = \frac{2 \pm \sqrt{4+4}}{2}$

$\Rightarrow \cot\theta = \frac{2 \pm 2\sqrt{2}}{2}$

$\Rightarrow \cot\theta = 1 \pm \sqrt{2}$

cot θ < 0 as θ is in 1^{st} quadrant.

So,

$\cot\theta = 1 + \sqrt{2}$

$\text{As, } 2\theta = 45° \Rightarrow \theta = \frac{45°}{2} = \frac{\pi}{8}$

$\Rightarrow \cot\frac{\pi}{8} = 1 + \sqrt{2}$

Hence Proved

28 A. Question

If $\cos x = -\frac{3}{5}$ and x lies in the IIIrd quadrant, find the values of $\cos\frac{x}{2}, \sin\frac{x}{2}$ and sin 2x.

Answer

Given:

$\cos x = -\frac{3}{5}$ and x lies in 3^{rd} quadrant $\Rightarrow x \in \left(\pi, \frac{3\pi}{2}\right)$

To find: Values of $\cos\frac{x}{2}, \sin\frac{x}{2}, \sin 2x$

We know,

$\cos 2x = 2\cos^2 x - 1$

$\Rightarrow \cos x = 2\cos^2\frac{x}{2} - 1$

$\Rightarrow -\frac{3}{5} = 2\cos^2\frac{x}{2} - 1$

$\left\{\because \cos x = -\frac{3}{5}\right\}$

$\Rightarrow 2\cos^2\frac{x}{2} = -\frac{3}{5} + 1$

$\Rightarrow 2\cos^2\frac{x}{2} = \frac{2}{5}$

$$\Rightarrow \cos^2\frac{x}{2} = \frac{1}{5}$$

$$\Rightarrow \cos\frac{x}{2} = \pm\frac{1}{\sqrt{5}}$$

Since,

$$x \in \left(\pi, \frac{3\pi}{2}\right) \Rightarrow \frac{x}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$

$\Rightarrow \cos\frac{x}{2}$ will be negative in third quadrant

So,

$$\cos x = -\frac{1}{\sqrt{5}}$$

We know,

cos 2x = 1 - 2 $\sin^2$ x

$$\Rightarrow \cos x = 1 - 2\sin^2\frac{x}{2}$$

$$\left\{\because \cos x = -\frac{3}{5}\right\}$$

$$\Rightarrow -\frac{3}{5} = 1 - 2\sin^2\frac{x}{2}$$

$$\Rightarrow 2\sin^2\frac{x}{2} = \frac{3}{5} + 1$$

$$\Rightarrow 2\sin^2\frac{x}{2} = \frac{8}{5}$$

$$\Rightarrow \sin^2\frac{x}{2} = \frac{4}{5}$$

$$\Rightarrow \sin\frac{x}{2} = \pm\frac{2}{\sqrt{5}}$$

Since,

$$x \in \left(\pi, \frac{3\pi}{2}\right) \Rightarrow \frac{x}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$

$\Rightarrow \sin\frac{x}{2}$ will be positive in second quadrant

So,

$$\Rightarrow \sin\frac{x}{2} = \frac{2}{\sqrt{5}}$$

We know,

$\sin^2$ x + $\cos^2$ x = 1

⇒ $\sin^2$ x = 1 - $\cos^2$ x

$$\Rightarrow \sin^2 x = 1 - \left(-\frac{3}{5}\right)^2$$

$$\left\{\because \cos x = -\frac{3}{5}\right\}$$

$\Rightarrow \sin^2 x = 1 - \frac{9}{25}$

$\Rightarrow \sin^2 x = \frac{25-9}{25}$

$\Rightarrow \sin^2 x = \frac{16}{25}$

$\Rightarrow \sin x = \pm\frac{4}{5}$

Since,

$x \in \left(\pi, \frac{3\pi}{2}\right)$

$\Rightarrow \sin x$ will be negative in third quadrant

So,

$\Rightarrow \sin x = -\frac{4}{5}$

Now,

sin 2x = 2(sin x)(cos x)

$\left\{\because \cos x = -\frac{3}{5} \,\&\, \sin x = -\frac{4}{5}\right\}$

$\Rightarrow \sin 2x = 2 \times -\frac{4}{5} \times -\frac{3}{5}$

$\Rightarrow \sin 2x = \frac{24}{25}$

Hence, values of $\cos\frac{x}{2}, \sin\frac{x}{2}, \sin 2x$ are $-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}$ and $\frac{24}{25}$

28 B. Question

If $\cos x = -\frac{3}{5}$ and x lies in the IInd quadrant, find the values of sin 2x and $\sin\frac{x}{2}$.

Answer

Given:

$\cos x = -\frac{3}{5}$ and x lies in 2^{nd} quadrant $\Rightarrow x \in \left(\frac{\pi}{2}, \pi\right)$

To find: Values of $\sin\frac{x}{2}$, sin 2x

We know,

cos 2x = 1 – 2 $\sin^2$ x

$\Rightarrow \cos x = 1 - 2\sin^2\frac{x}{2}$

$\left\{\because \cos x = -\frac{3}{5}\right\}$

$\Rightarrow -\frac{3}{5} = 1 - 2\sin^2\frac{x}{2}$

$\Rightarrow 2\sin^2\frac{x}{2} = \frac{3}{5} + 1$

$\Rightarrow 2\sin^2\frac{x}{2} = \frac{8}{5}$

$\Rightarrow \sin^2\frac{x}{2} = \frac{4}{5}$

$\Rightarrow \sin\frac{x}{2} = \pm\frac{2}{\sqrt{5}}$

Since,

$x \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \frac{x}{2} \in \left(\frac{\pi}{4}, \frac{3\pi}{2}\right)$

$\Rightarrow \sin\frac{x}{2}$ will be positive in first quadrant

So,

$\Rightarrow \sin\frac{x}{2} = \frac{2}{\sqrt{5}}$

We know,

$\sin^2 x + \cos^2 x = 1$

$\Rightarrow \sin^2 x = 1 - \cos^2 x$

$\Rightarrow \sin^2 x = 1 - \left(-\frac{3}{5}\right)^2$

$\left\{\because \cos x = -\frac{3}{5}\right\}$

$\Rightarrow \sin^2 x = 1 - \frac{9}{25}$

$\Rightarrow \sin^2 x = \frac{25 - 9}{25}$

$\Rightarrow \sin^2 x = \frac{16}{25}$

$\Rightarrow \sin x = \pm\frac{4}{5}$

Since,

$x \in \left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow$sin x will be positive in second quadrant

So,

$\Rightarrow \sin x = \frac{4}{5}$

Now,

sin 2x = 2(sin x)(cos x)

$\left\{\because \cos x = -\frac{3}{5} \;\&\; \sin x = \frac{4}{5}\right\}$

$\Rightarrow \sin 2x = 2 \times \frac{4}{5} \times -\frac{3}{5}$

$\Rightarrow \sin 2x = -\frac{24}{25}$

Hence, values of $\sin\frac{x}{2}$, $\sin 2x$ are $\frac{2}{\sqrt{5}}$ and $-\frac{24}{25}$

29. Question

If $\sin x = \frac{\sqrt{5}}{3}$ and x lies in IInd quadrant, find the values of $\cos\frac{x}{2}, \sin\frac{x}{2}$ and $\tan\frac{x}{2}$.

Answer

Given:

$\sin x = \frac{\sqrt{5}}{3}$ and x lies in 2^{nd} quadrant $\Rightarrow x \in \left(\frac{\pi}{2}, \pi\right)$

To find: Values of $\cos\frac{x}{2}, \sin\frac{x}{2}, \tan\frac{x}{2}$

We know,

$\sin^2 x + \cos^2 x = 1$

$\Rightarrow \cos^2 x = 1 - \sin^2 x$

$\Rightarrow \cos^2 x = 1 - \left(\frac{\sqrt{5}}{3}\right)^2$

$\left\{\because \sin x = \frac{\sqrt{5}}{3}\right\}$

$\Rightarrow \cos^2 x = 1 - \frac{5}{9}$

$\Rightarrow \cos^2 x = \frac{9-5}{9}$

$\Rightarrow \cos^2 x = \frac{4}{9}$

$\Rightarrow \cos x = \pm\frac{2}{3}$

Since,

$x \in \left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow$cosx will be negative in second quadrant

So,

$\Rightarrow \cos x = -\frac{2}{3}$

We know,

$\cos 2x = 2\cos^2 x - 1$

$\Rightarrow \cos x = 2\cos^2\frac{x}{2} - 1$

$\Rightarrow -\frac{2}{3} = 2\cos^2\frac{x}{2} - 1$

$\left\{\because \cos x = -\frac{2}{3}\right\}$

$\Rightarrow 2\cos^2\frac{x}{2} = -\frac{2}{3} + 1$

$\Rightarrow 2\cos^2\frac{x}{2} = \frac{-2+3}{3}$

$\Rightarrow \cos^2\frac{x}{2} = \frac{1}{6}$

$\Rightarrow \cos\frac{x}{2} = \pm\frac{1}{\sqrt{6}}$

Since,

$x \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \frac{x}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$\Rightarrow \cos\frac{x}{2}$ will be positive in first quadrant

So,

$\cos\frac{x}{2} = \frac{1}{\sqrt{6}}$

We know,

$\cos 2x = 1 - 2\sin^2 x$

$\Rightarrow \cos x = 1 - 2\sin^2\frac{x}{2}$

$\left\{\because \cos x = -\frac{2}{3}\right\}$

$\Rightarrow -\frac{2}{3} = 1 - 2\sin^2\frac{x}{2}$

$\Rightarrow 2\sin^2\frac{x}{2} = \frac{2}{3} + 1$

$\Rightarrow 2\sin^2\frac{x}{2} = \frac{2+3}{3}$

$\Rightarrow \sin^2\frac{x}{2} = \frac{5}{6}$

$\Rightarrow \sin\frac{x}{2} = \pm\sqrt{\frac{5}{6}}$

Since,

$x \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \frac{x}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$\Rightarrow \sin\frac{x}{2}$ will be positive in first quadrant

So,

$\Rightarrow \sin\frac{x}{2} = \sqrt{\frac{5}{6}}$

We know,

$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$

$\Rightarrow \tan\frac{x}{2} = \frac{\sqrt{\frac{5}{6}}}{\frac{1}{\sqrt{6}}}$

$\Rightarrow \tan\frac{x}{2} = \sqrt{5}$

Hence, values of $\cos\frac{x}{2}, \sin\frac{x}{2}, \tan\frac{x}{2}$ are $\frac{1}{\sqrt{6}}, \sqrt{\frac{5}{6}}$ and $\sqrt{5}$

30 A. Question

0 ≤ x ≤ π and x lies in the IInd quadrant such that $\sin x = \frac{1}{4}$. Find the values of $\cos\frac{x}{2}, \sin\frac{x}{2}$ and $\tan\frac{x}{2}$.

Answer

Given:

$\sin x = \frac{1}{4}$ and x lies in 2^{nd} quadrant $\Rightarrow x \in \left(\frac{\pi}{2}, \pi\right)$

To find: Values of $\cos\frac{x}{2}, \sin\frac{x}{2}, \tan\frac{x}{2}$

We know,

$\sin^2 x + \cos^2 x = 1$

$\Rightarrow \cos^2 x = 1 - \sin^2 x$

$\Rightarrow \cos^2 x = 1 - \left(\frac{1}{4}\right)^2$

$\left\{\because \sin x = \frac{1}{4}\right\}$

$\Rightarrow \cos^2 x = 1 - \frac{1}{16}$

$\Rightarrow \cos^2 x = \frac{16-1}{16}$

$\Rightarrow \cos^2 x = \frac{15}{16}$

$\Rightarrow \cos x = \pm\frac{\sqrt{15}}{4}$

Since,

$x \in \left(\frac{\pi}{2}, \pi\right)$

⇒cosx will be negative in second quadrant

So,

$$\Rightarrow \cos x = -\frac{\sqrt{15}}{4}$$

We know,

$\cos 2x = 2\cos^2 x - 1$

$$\Rightarrow \cos x = 2\cos^2\frac{x}{2} - 1$$

$$\Rightarrow -\frac{\sqrt{15}}{4} = 2\cos^2\frac{x}{2} - 1$$

$$\left\{\because \cos x = -\frac{\sqrt{15}}{4}\right\}$$

$$\Rightarrow 2\cos^2\frac{x}{2} = -\frac{\sqrt{15}}{4} + 1$$

$$\Rightarrow 2\cos^2\frac{x}{2} = \frac{-\sqrt{15}+4}{4}$$

$$\Rightarrow \cos^2\frac{x}{2} = \frac{-\sqrt{15}+4}{8}$$

$$\Rightarrow \cos\frac{x}{2} = \pm\sqrt{\frac{-\sqrt{15}+4}{8}}$$

Since,

$$x \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \frac{x}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$\Rightarrow \cos\frac{x}{2}$ will be positive in first quadrant

So,

$$\cos\frac{x}{2} = \sqrt{\frac{-\sqrt{15}+4}{8}}$$

We know,

$\cos 2x = 1 - 2\sin^2 x$

$$\Rightarrow \cos x = 1 - 2\sin^2\frac{x}{2}$$

$$\left\{\because \cos x = -\frac{\sqrt{15}}{4}\right\}$$

$$\Rightarrow -\frac{\sqrt{15}}{4} = 1 - 2\sin^2\frac{x}{2}$$

$$\Rightarrow 2\sin^2\frac{x}{2} = \frac{\sqrt{15}}{4} + 1$$

$$\Rightarrow 2\sin^2\frac{x}{2} = \frac{\sqrt{15}+4}{4}$$

$$\Rightarrow \sin^2\frac{x}{2} = \frac{\sqrt{15}+4}{8}$$

$$\Rightarrow \sin\frac{x}{2} = \pm\sqrt{\frac{\sqrt{15}+4}{8}}$$

Since,

$$x \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \frac{x}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$\Rightarrow \sin\frac{x}{2}$ will be positive in first quadrant

So,

$$\Rightarrow \sin\frac{x}{2} = \sqrt{\frac{\sqrt{15}+4}{8}}$$

We know,

$$\tan\frac{x}{2} = \frac{\sqrt{\frac{\sqrt{15}+4}{8}}}{\sqrt{\frac{-\sqrt{15}+4}{8}}}$$

$$\Rightarrow \tan\frac{x}{2} = \sqrt{\frac{\sqrt{15}+4}{8} \times \frac{8}{-\sqrt{15}+4}}$$

$$\Rightarrow \tan\frac{x}{2} = \sqrt{\frac{\sqrt{15}+4}{-\sqrt{15}+4}}$$

On rationalising:

$$\Rightarrow \tan\frac{x}{2} = \sqrt{\frac{4+\sqrt{15}}{4-\sqrt{15}} \times \frac{4+\sqrt{15}}{4+\sqrt{15}}}$$

$$\Rightarrow \tan\frac{x}{2} = \sqrt{\frac{\left(4+\sqrt{15}\right)^2}{4^2-\left(\sqrt{15}\right)^2}}$$

$\{\because (a + b)(a - b) = a^2 - b^2\}$

$$\Rightarrow \tan\frac{x}{2} = \sqrt{\frac{\left(4+\sqrt{15}\right)^2}{16-15}}$$

$$\Rightarrow \tan\frac{x}{2} = \sqrt{\frac{\left(4+\sqrt{15}\right)^2}{1}}$$

$$\Rightarrow \tan\frac{x}{2} = 4+\sqrt{15}$$

Hence, values of $\cos\frac{x}{2}, \sin\frac{x}{2}, \tan\frac{x}{2}$ are $\sqrt{\frac{-\sqrt{15}+4}{8}}, \sqrt{\frac{\sqrt{15}+4}{8}}$ and $4+\sqrt{15}$

30 B. Question

If $\cos x = \frac{4}{5}$ and x is acute, find tan 2x.

Answer

Given:

$\cos x = \frac{4}{5}$ and x is acute $\Rightarrow x \in \left(0, \frac{\pi}{2}\right)$

To find: Value of tan 2x

We know,

$\sin^2 x + \cos^2 x = 1$

$\Rightarrow \sin^2 x = 1 - \cos^2 x$

$\Rightarrow \sin^2 x = 1 - \left(\frac{4}{5}\right)^2$

$\left\{\because \cos x = \frac{4}{5}\right\}$

$\Rightarrow \sin^2 x = 1 - \frac{16}{25}$

$\Rightarrow \sin^2 x = \frac{25 - 16}{25}$

$\Rightarrow \sin^2 x = \frac{9}{25}$

$\Rightarrow \sin x = \pm\frac{3}{5}$

Since,

$x \in \left(0, \frac{\pi}{2}\right)$

⇒sinx will be negative in first quadrant

So,

$\Rightarrow \sin x = \frac{3}{5}$

Now,

$\tan x = \frac{\sin x}{\cos x}$

$\Rightarrow \tan x = \frac{\frac{3}{5}}{\frac{4}{5}}$

$\Rightarrow \tan x = \frac{3}{4}$

We know,

$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

$$\Rightarrow \tan 2x = \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}$$

$$\Rightarrow \tan 2x = \frac{\frac{3}{2}}{1-\frac{9}{16}}$$

$$\Rightarrow \tan 2x = \frac{\frac{3}{2}}{\frac{16-9}{16}}$$

$$\Rightarrow \tan 2x = \frac{\frac{3}{2}}{\frac{7}{16}}$$

$$\Rightarrow \tan 2x = \frac{3}{2} \times \frac{16}{7}$$

$$\Rightarrow \tan 2x = \frac{24}{7}$$

Hence, value of $\tan 2x = \frac{24}{7}$

30 C. Question

If $\sin x = \frac{4}{5}$ and $0 < x < \frac{\pi}{2}$, find the value of sin 4x.

Answer

Given:

$$\sin x = \frac{4}{5} \text{ and } x \in \left(0, \frac{\pi}{2}\right)$$

To find: Values of sin4x

We know,

$\sin^2 x + \cos^2 x = 1$

$\Rightarrow \cos^2 x = 1 - \sin^2 x$

$$\Rightarrow \cos^2 x = 1 - \left(\frac{4}{5}\right)^2$$

$$\left\{\because \sin x = \frac{4}{5}\right\}$$

$$\Rightarrow \cos^2 x = 1 - \frac{16}{25}$$

$$\Rightarrow \cos^2 x = \frac{25-16}{25}$$

$$\Rightarrow \cos^2 x = \frac{9}{25}$$

$$\Rightarrow \cos x = \pm\frac{3}{5}$$

Since,

$x \in \left(0, \frac{\pi}{2}\right)$

⇒cosx will be negative in first quadrant

So,

$\Rightarrow \cos x = \frac{3}{5}$

We know,

sin 2x = 2 sin x cos x

cos 2x = 2 $\cos^2$ x – 1

Therefore,

sin 4x = 2 sin 2x cos 2x

⇒ sin 4x = 2 (2 sin x cos x) (2 $\cos^2$ x – 1)

$\left\{\because \sin x = \frac{4}{5} \& \cos x = \frac{4}{5}\right\}$

$$\Rightarrow \sin 4x = 2\left(2 \times \frac{4}{5} \times \frac{3}{5}\right)\left(2\left(\frac{4}{5}\right)^2 - 1\right)$$

$$\Rightarrow \sin 4x = 2\left(\frac{24}{25}\right)\left(2 \times \frac{16}{25} - 1\right)$$

$$\Rightarrow \sin 4x = \frac{48}{25}\left(\frac{32}{25} - 1\right)$$

$$\Rightarrow \sin 4x = \frac{48}{25}\left(\frac{32 - 25}{25}\right)$$

$$\Rightarrow \sin 4x = \frac{48}{25}\left(\frac{7}{25}\right)$$

$$\Rightarrow \sin 4x = \frac{336}{625}$$

Hence, value of $\sin 4x = \frac{336}{625}$

31. Question

If $\tan x = \frac{b}{a}$, then find the value of $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$.

Answer

Given: $\tan x = \frac{b}{a}$

To find: $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$

$$\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$$

On taking LCM:

$$= \frac{(\sqrt{a+b})^2 + (\sqrt{a-b})^2}{\sqrt{a+b}\sqrt{a-b}}$$

$$= \frac{a+b+a-b}{\sqrt{a+b}\sqrt{a-b}}$$

$$= \frac{2a}{\sqrt{a+b}\sqrt{a-b}}$$

Dividing numerator and denominator by a:

$$= \frac{\frac{2a}{a}}{\frac{\sqrt{a+b}\sqrt{a-b}}{a}}$$

$$= \frac{2}{\sqrt{\frac{a+b}{a}}\sqrt{\frac{a-b}{a}}}$$

$$= \frac{2}{\sqrt{1+\frac{b}{a}}\sqrt{1-\frac{b}{a}}}$$

$$= \frac{2}{\sqrt{1+\tan x}\sqrt{1-\tan x}}$$

$\left\{\because \tan x = \frac{b}{a}\right\}$

$$= \frac{2}{\sqrt{(1+\tan x)(1-\tan x)}}$$

$\{\because (a + b)(a - b) = a^2 - b^2\}$

$$= \frac{2}{\sqrt{1-\tan^2 x}}$$

32. Question

If $\tan A = \frac{1}{7}$ and $\tan B = \frac{1}{3}$, show that cos 2A = sin 4B

Answer

Given: $\tan A = \frac{1}{7}$ & $\tan B = \frac{1}{3}$

To prove: cos 2A = sin 4B

We know,

$$\tan 2B = \frac{2\tan B}{1-\tan^2 B}$$

$$\Rightarrow \tan 2B = \frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}$$

$$\Rightarrow \tan 2B = \frac{\frac{2}{3}}{1-\frac{1}{9}}$$

$$\Rightarrow \tan 2B = \frac{\frac{2}{3}}{\frac{9-1}{9}}$$

$$\Rightarrow \tan 2B = \frac{\frac{2}{3}}{\frac{8}{9}}$$

$$\Rightarrow \tan 2B = \frac{3}{4}$$

Take LHS:

cos 2A

$$= \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\left\{\because \tan A = \frac{1}{7}\right\}$$

$$= \frac{1-\left(\frac{1}{7}\right)^2}{1+\left(\frac{1}{7}\right)^2}$$

$$= \frac{1-\frac{1}{49}}{1+\frac{1}{49}}$$

$$= \frac{\frac{49-1}{49}}{\frac{49+1}{49}}$$

$$= \frac{\frac{48}{49}}{\frac{50}{49}}$$

$$= \frac{48}{50}$$

$$= \frac{24}{25}$$

Now,

Take RHS:

sin 4B

$$= \frac{2\tan 2B}{1+\tan^2 2B}$$

$$\left\{\because \tan 2B = \frac{3}{4}\right\}$$

$$= \frac{2\left(\frac{3}{4}\right)}{1+\left(\frac{3}{4}\right)^2}$$

$$= \frac{\frac{3}{2}}{1+\frac{9}{16}}$$

$= \frac{\frac{3}{2}}{\frac{16+9}{16}}$

$= \frac{\frac{3}{2}}{\frac{25}{16}}$

$= \frac{24}{25}$

Clearly, LHS = RHS $= \frac{24}{25}$

Hence Proved

33. Question

Prove that:

$$\cos 7^{\circ} \cos 14^{\circ} \cos 28^{\circ} \cos 56^{\circ} = \frac{\sin 68^{\circ}}{16 \cos 83^{\circ}}$$

Answer

To prove: $\cos 7^{\circ} \cos 14^{\circ} \cos 28^{\circ} \cos 56^{\circ} = \frac{\sin 68^{\circ}}{16 \cos 83^{\circ}}$

Proof:

Take LHS:

$\cos 7^{\circ} \cos 14^{\circ} \cos 28^{\circ} \cos 56^{\circ}$

Multiplying and Dividing $2^4 \sin 7^{\circ}$

$= \frac{2^4 \sin 7^{\circ} \cos 7^{\circ} \cos 14^{\circ} \cos 28^{\circ} \cos 56^{\circ}}{2^4 \sin 7^{\circ}}$

$= \frac{2^3 (2 \sin 7^{\circ} \cos 7^{\circ}) \cos 14^{\circ} \cos 28^{\circ} \cos 56^{\circ}}{2^4 \sin 7^{\circ}}$

{∵ sin 2x = 2 sin x cos x}

$= \frac{2^3 (\sin 14^{\circ}) \cos 14^{\circ} \cos 28^{\circ} \cos 56^{\circ}}{2^4 \sin 7^{\circ}}$

$= \frac{2^2 (2 \sin 14^{\circ} \cos 14^{\circ}) \cos 28^{\circ} \cos 56^{\circ}}{2^4 \sin 7^{\circ}}$

$= \frac{2^2 (\sin 28^{\circ}) \cos 28^{\circ} \cos 56^{\circ}}{2^4 \sin 7^{\circ}}$

$= \frac{2^1 (2 \sin 28^{\circ} \cos 28^{\circ}) \cos 56^{\circ}}{2^4 \sin 7^{\circ}}$

$= \frac{2^1 (\sin 56^{\circ}) \cos 56^{\circ}}{2^4 \sin 7^{\circ}}$

$= \frac{2 \sin 56^{\circ} \cos 56^{\circ}}{2^4 \sin 7^{\circ}}$

$= \frac{\sin 112^{\circ}}{2^4 \sin 7^{\circ}}$

We know,

sin (180° - θ) = sin θ

sin (90° - θ) = cos θ

Now,

$$= \frac{\sin(180° - 112°)}{2^4 \cos(90° - 7°)}$$

$$= \frac{\sin 68°}{16 \cos 83°}$$

= RHS

Hence Proved

34. Question

Prove that:

$$\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} = \frac{1}{16}$$

Answer

To prove: $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} = \frac{1}{16}$

Proof:

Take LHS:

$$\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}$$

Multiplying and Dividing by $2^4 \sin\frac{2\pi}{15}$:

$$= \frac{2^4 \sin\frac{2\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}}{2^4 \sin\frac{2\pi}{15}}$$

$$= \frac{2^3 \left(2\sin\frac{2\pi}{15}\cos\frac{2\pi}{15}\right)\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}}{2^4 \sin\frac{2\pi}{15}}$$

{∵ sin 2x = 2 sin x cos x}

$$= \frac{2^3 \sin\frac{4\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}}{2^4 \sin\frac{2\pi}{15}}$$

$$= \frac{2^2 \left(2\sin\frac{4\pi}{15}\cos\frac{4\pi}{15}\right)\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}}{2^4 \sin\frac{2\pi}{15}}$$

$$= \frac{2^2 \sin\frac{8\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}}{2^4 \sin\frac{2\pi}{15}}$$

$$= \frac{2\left(2\sin\frac{8\pi}{15}\cos\frac{8\pi}{15}\right)\cos\frac{16\pi}{15}}{2^4 \sin\frac{2\pi}{15}}$$

$$= \frac{2 \sin \frac{16\pi}{15} \cos \frac{16\pi}{15}}{2^4 \sin \frac{2\pi}{15}}$$

$$= \frac{\sin \frac{32\pi}{15}}{2^4 \sin \frac{2\pi}{15}}$$

$$= \frac{\sin \left(2\pi + \frac{2\pi}{15}\right)}{2^4 \sin \frac{2\pi}{15}}$$

$$\left\{\because 2\pi + \frac{2\pi}{15} = \frac{30\pi + 2\pi}{15} = \frac{32\pi}{15}\right\}$$

$$= \frac{\sin \frac{2\pi}{15}}{2^4 \sin \frac{2\pi}{15}}$$

{∵ sin (2π + θ) = sin θ}

$$= \frac{1}{2^4}$$

$$= \frac{1}{16}$$

= RHS

Hence Proved

35. Question

Prove that:

$$\cos \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \cos \frac{8\pi}{5} = \frac{-1}{16}$$

Answer

To prove: $\cos \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \cos \frac{8\pi}{5} = \frac{-1}{16}$

Proof:

Take LHS:

$$\cos \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \cos \frac{8\pi}{5}$$

Multiplying and Dividing $2^4 \sin \frac{\pi}{5}$:

$$= \frac{2^4 \sin \frac{\pi}{5} \cos \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \cos \frac{8\pi}{5}}{2^4 \sin \frac{\pi}{5}}$$

$$= \frac{2^3 \left(2 \sin \frac{\pi}{5} \cos \frac{\pi}{5}\right) \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \cos \frac{8\pi}{5}}{2^4 \sin \frac{\pi}{5}}$$

{∵ sin 2x = 2 sin x cos x}

$$= \frac{2^3 \sin\frac{2\pi}{5}\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5}}{2^4 \sin\frac{\pi}{5}}$$

$$= \frac{2^2 \left(2\sin\frac{2\pi}{5}\cos\frac{2\pi}{5}\right)\cos\frac{4\pi}{5}\cos\frac{8\pi}{5}}{2^4 \sin\frac{\pi}{5}}$$

$$= \frac{2^2 \sin\frac{4\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5}}{2^4 \sin\frac{\pi}{5}}$$

$$= \frac{2\left(2\sin\frac{4\pi}{5}\cos\frac{4\pi}{5}\right)\cos\frac{8\pi}{5}}{2^4 \sin\frac{\pi}{5}}$$

$$= \frac{2\sin\frac{8\pi}{5}\cos\frac{8\pi}{5}}{2^4 \sin\frac{\pi}{5}}$$

$$= \frac{\sin\frac{16\pi}{5}}{2^4 \sin\frac{\pi}{5}}$$

$$= \frac{\sin\left(3\pi + \frac{\pi}{5}\right)}{2^4 \sin\frac{\pi}{5}}$$

$$\left\{\because 3\pi + \frac{\pi}{5} = \frac{15\pi + \pi}{5} = \frac{16\pi}{5}\right\}$$

$$= -\frac{\sin\frac{\pi}{5}}{2^4 \sin\frac{\pi}{5}}$$

{∵ sin (3π + θ) = - sin θ}

$$= -\frac{1}{2^4}$$

$$= -\frac{1}{16}$$

= RHS

Hence Proved

36. Question

Prove that:

$$\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}$$
$$\cos\frac{32\pi}{65} = \frac{1}{64}$$

Answer

To prove: $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65} = \frac{1}{64}$

Proof:

Take LHS:

$$\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}$$

Multiplying and Dividing $2^6 \sin\frac{\pi}{65}$:

$$=\frac{2^6\sin\frac{\pi}{65}\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}}{2^6\sin\frac{\pi}{65}}$$

$$=\frac{2^5\left(2\sin\frac{\pi}{65}\cos\frac{\pi}{65}\right)\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}}{2^6\sin\frac{\pi}{65}}$$

{∵ sin 2x = 2 sin x cos x}

$$=\frac{2^5\sin\frac{2\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}}{2^6\sin\frac{\pi}{65}}$$

$$=\frac{2^4\left(2\sin\frac{2\pi}{65}\cos\frac{2\pi}{65}\right)\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}}{2^6\sin\frac{\pi}{65}}$$

$$=\frac{2^4\sin\frac{4\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}}{2^6\sin\frac{\pi}{65}}$$

$$=\frac{2^3\left(2\sin\frac{4\pi}{65}\cos\frac{4\pi}{65}\right)\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}}{2^6\sin\frac{\pi}{65}}$$

$$=\frac{2^3\sin\frac{8\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}}{2^6\sin\frac{\pi}{65}}$$

$$=\frac{2^2\left(2\sin\frac{8\pi}{65}\cos\frac{8\pi}{65}\right)\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}}{2^6\sin\frac{\pi}{65}}$$

$$=\frac{2^2\sin\frac{16\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}}{2^6\sin\frac{\pi}{65}}$$

$$=\frac{2\left(2\sin\frac{16\pi}{65}\cos\frac{16\pi}{65}\right)\cos\frac{32\pi}{65}}{2^6\sin\frac{\pi}{65}}$$

$$=\frac{2\sin\frac{32\pi}{65}\cos\frac{32\pi}{65}}{2^6\sin\frac{\pi}{65}}$$

$$=\frac{\sin\frac{64\pi}{65}}{2^6\sin\frac{\pi}{65}}$$

$$=\frac{\sin\left(\pi-\frac{\pi}{65}\right)}{2^6\sin\frac{\pi}{65}}$$

$\left\{\because \pi - \frac{\pi}{65} = \frac{65\pi - \pi}{65} = \frac{64\pi}{65}\right\}$

$= \frac{\sin\frac{\pi}{65}}{2^5 \sin\frac{\pi}{65}}$

{∵ sin (π - θ) = sin θ}

$= \frac{1}{2^5}$

$= \frac{1}{64}$

= RHS

Hence Proved

37. Question

If 2 tan α = 3 tan β, prove that tan (α - β) $= \frac{\sin 2\beta}{5 - \cos 2\beta}$.

Answer

Given: 2 tan α = 3 tan β

To prove: $\tan(\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}$

Proof:

Take LHS:

tan α – tan β

$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$= \frac{\frac{3}{2}\tan\beta - \tan\beta}{1 + \frac{3}{2}\tan\beta\tan\beta}$

$\left\{\because 2\tan\alpha = 3\tan\beta \Rightarrow \tan\alpha = \frac{3}{2}\tan\beta\right\}$

$= \frac{\tan\beta\left(\frac{3}{2} - 1\right)}{1 + \frac{3}{2}\tan^2\beta}$

$= \frac{\frac{1}{2}\tan\beta}{1 + \frac{3}{2}\tan^2\beta}$

$= \frac{\frac{1}{2}\frac{\sin\beta}{\cos\beta}}{1 + \frac{3}{2}\cdot\left(\frac{\sin\beta}{\cos\beta}\right)^2}$

$\left\{\because \tan\beta = \frac{\sin\beta}{\cos\beta}\right\}$

$$= \frac{\frac{\sin\beta}{2\cos\beta}}{1 + \frac{3\sin^2\beta}{2\cos^2\beta}}$$

$$= \frac{\frac{\sin\beta}{2\cos\beta}}{\frac{2\cos^2\beta + 3\sin^2\beta}{2\cos^2\beta}}$$

$$= \frac{2\cos^2\beta\sin\beta}{2\cos\beta\,(2\cos^2\beta + 3\sin^2\beta)}$$

$$= \frac{2\cos\beta\sin\beta}{2(2\cos^2\beta + 3\sin^2\beta)}$$

$$= \frac{\sin 2\beta}{2(2\cos^2\beta) + 3(2\sin^2\beta)}$$

{∵ sin 2x = 2(sin x)(cos x)}

$$= \frac{\sin 2\beta}{2(1 + \cos 2\beta) + 3(1 - \cos 2\beta)}$$

{∵ 2 $\cos^2$ x = 1 + cos 2x & 2 $\sin^2$ x = 1 - cos 2x}

$$= \frac{\sin 2\beta}{2 + 2\cos 2\beta + 3 - 3\cos 2\beta}$$

$$= \frac{\sin 2\beta}{5 - \cos 2\beta}$$

= RHS

Hence Proved

38 A. Question

If sin α + sin β = a and cos α + cos β = b, prove that

$$\sin(\alpha + \beta) = \frac{2ab}{a^2 + b^2}$$

Answer

Given: sin α + sin β = a & cos α + cos β = b

To prove: $\sin(\alpha + \beta) = \frac{2ab}{a^2 + b^2}$

Proof:

sin α + sin β = a ....(3)

cos α + cos β = b ......(4)

Dividing equation 3 and 4:

$$\Rightarrow \frac{(\sin\alpha + \sin\beta)}{(\cos\alpha + \cos\beta)} = \frac{a}{b}$$

$$\Rightarrow \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}} = \frac{a}{b}$$

$$\Rightarrow \frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2}} = \frac{a}{b}$$

$$\Rightarrow \tan\frac{\alpha+\beta}{2} = \frac{a}{b}$$

We know,

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

Therefore,

$$\sin(\alpha+\beta) = \frac{2\tan\frac{\alpha+\beta}{2}}{1+\tan^2\frac{\alpha+\beta}{2}}$$

$$\Rightarrow \sin(\alpha+\beta) = \frac{2\left(\frac{a}{b}\right)}{1+\left(\frac{a}{b}\right)^2}$$

$$\Rightarrow \sin(\alpha+\beta) = \frac{\frac{2a}{b}}{1+\frac{a^2}{b^2}}$$

$$\Rightarrow \sin(\alpha+\beta) = \frac{\frac{2a}{b}}{\frac{b^2+a^2}{b^2}}$$

$$\Rightarrow \sin(\alpha+\beta) = \frac{\frac{2a}{1}}{\frac{b^2+a^2}{b}}$$

$$\Rightarrow \sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}$$

Hence Proved

38 B. Question

If sin α + sin β = a and cos α + cos β = b, prove that

$$\cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}$$

Answer

Given: sin α + sin β = a & cos α + cos β = b

To prove: $\cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}$

Proof:

sin α + sin β = a

Squaring both sides, we get

$(\sin\alpha + \sin\beta)^2 = a^2$

$\Rightarrow \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta = a^2$ ……(1)

cos α + cos β = b

Squaring both sides, we get

$(\cos \alpha + \cos \beta)^2 = a^2$

$\Rightarrow \cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta = b^2$ ....(2)

Adding equation 1 and 2, we get

$\sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta + \cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta = a^2 + b^2$

$\Rightarrow \sin^2 \alpha + \cos^2 \alpha + \sin^2 \beta + \cos^2 \beta + 2 \sin \alpha \sin \beta + 2 \cos \alpha \cos \beta = a^2 + b^2$

$\Rightarrow 1 + 1 + 2 \sin \alpha \sin \beta + 2 \cos \alpha \cos \beta = a^2 + b^2$

$\{\because \sin^2 x + \cos^2 x = 1\}$

$\Rightarrow 2 + 2 \sin \alpha \sin \beta + 2 \cos \alpha \cos \beta = a^2 + b^2$

$\Rightarrow 2(\sin \alpha \sin \beta + \cos \alpha \cos \beta) = a^2 + b^2 - 2$

$$\Rightarrow (\sin \alpha \sin \beta + \cos \alpha \cos \beta) = \frac{a^2 + b^2 - 2}{2}$$

We know,

$\sin A \sin B + \cos A \cos B = \cos (A - B)$

Therefore,

$$\Rightarrow \cos(\alpha - \beta) = \frac{a^2 + b^2 - 2}{2}$$

Hence Proved

39. Question

If $2 \tan \frac{\alpha}{2} = \tan \frac{\beta}{2}$, prove that $\cos \alpha = \frac{3 + 5 \cos \beta}{5 + 3 \cos \beta}$.

Answer

Given: $2 \tan \frac{\alpha}{2} = \tan \frac{\beta}{2}$

To prove: $\cos \alpha = \frac{3 + 5 \cos \beta}{5 + 3 \cos \beta}$

Proof:

Take LHS:

$\cos \alpha$

$$= \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$$

$$\left\{\because \tan \frac{\alpha}{2} = \frac{1}{2} \tan \frac{\beta}{2}\right\}$$

$$= \frac{1 - \left(\frac{1}{2} \tan \frac{\beta}{2}\right)^2}{1 + \left(\frac{1}{2} \tan \frac{\beta}{2}\right)^2}$$

$$= \frac{1 - \frac{1}{4}\tan^2\frac{\beta}{2}}{1 + \frac{1}{4}\tan^2\frac{\beta}{2}}$$

$$= \frac{\frac{4 - \tan^2\frac{\beta}{2}}{4}}{\frac{4 + \tan^2\frac{\beta}{2}}{4}}$$

$$= \frac{4 - \tan^2\frac{\beta}{2}}{4 + \tan^2\frac{\beta}{2}}$$

Now, Take RHS:

$$\frac{3 + 5\cos\beta}{5 + 3\cos\beta}$$

$$= \frac{3 + 5\left(\frac{1 - \tan^2\frac{\beta}{2}}{1 + \tan^2\frac{\beta}{2}}\right)}{5 + 3\left(\frac{1 - \tan^2\frac{\beta}{2}}{1 + \tan^2\frac{\beta}{2}}\right)}$$

$$\left\{\because \cos x = \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right\}$$

$$= \frac{\frac{3\left(1 + \tan^2\frac{\beta}{2}\right) + 5\left(1 - \tan^2\frac{\beta}{2}\right)}{1 + \tan^2\frac{\beta}{2}}}{\frac{5\left(1 + \tan^2\frac{\beta}{2}\right) + 3\left(1 - \tan^2\frac{\beta}{2}\right)}{1 + \tan^2\frac{\beta}{2}}}$$

$$= \frac{3 + 3\tan^2\frac{\beta}{2} + 5 - 5\tan^2\frac{\beta}{2}}{5 + 5\tan^2\frac{\beta}{2} + 3 - 3\tan^2\frac{\beta}{2}}$$

$$= \frac{8 - 2\tan^2\frac{\beta}{2}}{8 + 2\tan^2\frac{\beta}{2}}$$

$$= \frac{2\left(4 - \tan^2\frac{\beta}{2}\right)}{2\left(4 + \tan^2\frac{\beta}{2}\right)}$$

$$= \frac{4 - \tan^2\frac{\beta}{2}}{4 + \tan^2\frac{\beta}{2}}$$

$$\left\{\because \cos\alpha == \frac{4 - \tan^2\frac{\beta}{2}}{4 + \tan^2\frac{\beta}{2}}\right\}$$

$= \cos\alpha$

Hence Proved

40. Question

If $\cos x = \dfrac{\cos\alpha + \cos\beta}{1 + \cos\alpha\cos\beta}$, prove that $\tan\dfrac{x}{2} = \pm\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$

Answer

Given: $\cos x = \dfrac{\cos\alpha + \cos\beta}{1 + \cos\alpha\cos\beta}$

To prove: $\tan\dfrac{x}{2} = \pm\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$

$$\cos x = \frac{\cos\alpha + \cos\beta}{1 + \cos\alpha\cos\beta}$$

We know,

$$\cos x = \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$$

$$\Rightarrow \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = \frac{\frac{1 - \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}} + \frac{1 - \tan^2\frac{\beta}{2}}{1 + \tan^2\frac{\beta}{2}}}{1 + \left(\frac{1 - \tan^2\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}}\right)\left(\frac{1 - \tan^2\frac{\beta}{2}}{1 + \tan^2\frac{\beta}{2}}\right)}$$

$$\Rightarrow \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = \frac{\frac{\left(1 - \tan^2\frac{\alpha}{2}\right)\left(1 + \tan^2\frac{\beta}{2}\right) + \left(1 - \tan^2\frac{\beta}{2}\right)\left(1 + \tan^2\frac{\alpha}{2}\right)}{\left(1 + \tan^2\frac{\alpha}{2}\right)\left(1 + \tan^2\frac{\beta}{2}\right)}}{\frac{\left(1 + \tan^2\frac{\alpha}{2}\right)\left(1 + \tan^2\frac{\beta}{2}\right) + \left(1 - \tan^2\frac{\alpha}{2}\right)\left(1 - \tan^2\frac{\beta}{2}\right)}{\left(1 + \tan^2\frac{\alpha}{2}\right)\left(1 + \tan^2\frac{\beta}{2}\right)}}$$

$$\Rightarrow \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = \frac{\left(1 - \tan^2\frac{\alpha}{2}\right)\left(1 + \tan^2\frac{\beta}{2}\right) + \left(1 - \tan^2\frac{\beta}{2}\right)\left(1 + \tan^2\frac{\alpha}{2}\right)}{\left(1 + \tan^2\frac{\alpha}{2}\right)\left(1 + \tan^2\frac{\beta}{2}\right) + \left(1 - \tan^2\frac{\alpha}{2}\right)\left(1 - \tan^2\frac{\beta}{2}\right)}$$

$$\Rightarrow \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$$

$$= \frac{1 - \tan^2\frac{\alpha}{2} + \tan^2\frac{\beta}{2} - \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2} + 1 - \tan^2\frac{\beta}{2} + \tan^2\frac{\alpha}{2} - \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}{1 + \tan^2\frac{\alpha}{2} + \tan^2\frac{\beta}{2} + \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2} + 1 - \tan^2\frac{\alpha}{2} - \tan^2\frac{\beta}{2} + \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}$$

$$\Rightarrow \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = \frac{2 - 2\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}{2 + 2\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}$$

$$\Rightarrow \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = \frac{2\left(1 - \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)}{2\left(1 + \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)}$$

$$\Rightarrow \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}$$

Applying componendo and dividendo, we get

$$\Rightarrow \frac{\left(1-\tan^2\frac{x}{2}\right)+\left(1+\tan^2\frac{x}{2}\right)}{\left(1-\tan^2\frac{x}{2}\right)-\left(1+\tan^2\frac{x}{2}\right)} = \frac{\left(1-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)+\left(1+\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)}{\left(1-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)-\left(1+\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)}$$

$$\Rightarrow \frac{1-\tan^2\frac{x}{2}+1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}-1-\tan^2\frac{x}{2}} = \frac{1-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}+1+\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}{1-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}-1-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}$$

$$\Rightarrow \frac{2}{-2\tan^2\frac{x}{2}} = \frac{2}{-2\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}$$

$$\Rightarrow \frac{1}{-\tan^2\frac{x}{2}} = \frac{1}{-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}$$

Taking reciprocal both sides:

$$\Rightarrow -\tan^2\frac{x}{2} = -\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}$$

$$\Rightarrow \tan^2\frac{x}{2} = \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}$$

$$\Rightarrow \tan\frac{x}{2} = \pm\sqrt{\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}}$$

$$\Rightarrow \tan\frac{x}{2} = \pm\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

Hence Proved

41. Question

If sec (x + α) + sec(x - α) = 2 sec x, prove that cos x$= \pm\sqrt{2}\cos\frac{\alpha}{2}$

Answer

Given: sec (x + α) + sec(x - α) = 2 sec x

To prove: $\cos x = \pm\sqrt{2}\cos\frac{\alpha}{2}$

sec (x + α) + sec(x - α) = 2 sec x

$$\Rightarrow \frac{1}{\cos(x+\alpha)} + \frac{1}{\cos(x-\alpha)} = \frac{2}{\cos x}$$

$$\left\{\because \sec x = \frac{1}{\cos x}\right\}$$

$$\Rightarrow \frac{\cos(x-\alpha)+\cos(x+\alpha)}{\cos(x+\alpha)\cos(x-\alpha)} = \frac{2}{\cos x}$$

$$\left\{\because \cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\right\}$$

$$\Rightarrow \frac{2\cos\left(\frac{x+\alpha+x-\alpha}{2}\right)\cos\left(\frac{x+\alpha-x+\alpha}{2}\right)}{\cos(x+\alpha)\cos(x-\alpha)} = \frac{2}{\cos x}$$

$$\Rightarrow \frac{2\cos\left(\frac{2x}{2}\right)\cos\left(\frac{2\alpha}{2}\right)}{2\cos(x+\alpha)\cos(x-\alpha)} = \frac{1}{\cos x}$$

{∵ 2 cos A cos B = cos (A + B) + cos (A – B)}

$$\Rightarrow \frac{2\cos x\cos\alpha}{\cos(x+\alpha+x-\alpha)+\cos(x+\alpha-x+\alpha)} = \frac{1}{\cos x}$$

$$\Rightarrow \frac{2\cos x\cos\alpha}{\cos 2x+\cos 2\alpha} = \frac{1}{\cos x}$$

$$\Rightarrow 2\cos^2 x\cos\alpha = \cos 2x + \cos 2\alpha$$

$$\Rightarrow 2\cos^2 x\cos\alpha = 2\cos^2 x - 1 + \cos 2\alpha$$

{∵ cos 2x = 2 $\cos^2$ x – 1}

$$\Rightarrow 2\cos^2 x\cos\alpha - 2\cos^2 x = \cos 2\alpha - 1$$

$$\Rightarrow 2\cos^2 x(\cos\alpha - 1) = 2\cos^2\alpha - 1 - 1$$

{∵ cos 2x = 2 $\cos^2$ x – 1}

$$\Rightarrow 2\cos^2 x = \frac{2\cos^2\alpha - 2}{\cos\alpha - 1}$$

$$\Rightarrow 2\cos^2 x = \frac{2(\cos^2\alpha - 1)}{\cos\alpha - 1}$$

$$\Rightarrow \cos^2 x = \frac{(\cos\alpha - 1)(\cos\alpha + 1)}{\cos\alpha - 1}$$

$$\Rightarrow \cos^2 x = \cos\alpha + 1$$

$$\Rightarrow \cos^2 x = 2\cos^2\frac{\alpha}{2} - 1 + 1$$

$$\left\{\because \cos x = 2\cos^2\frac{x}{2} - 1\right\}$$

$$\Rightarrow \cos^2 x = 2\cos^2\frac{\alpha}{2}$$

$$\Rightarrow \cos x = \pm\sqrt{2\cos^2\frac{\alpha}{2}}$$

$$\Rightarrow \cos x = \pm\sqrt{2}\cos\frac{\alpha}{2}$$

Hence Proved

42. Question

If $\cos\alpha + \cos\beta = \frac{1}{3}$ and $\sin\alpha + \sin\beta = \frac{1}{4}$, prove that $\cos\frac{\alpha-\beta}{2} = \pm\frac{5}{24}$.

Answer

Given: $\cos\alpha + \cos\beta = \frac{1}{3}$ & $\sin\alpha + \sin\beta = \frac{1}{4}$

To prove: $\cos\frac{\alpha-\beta}{2} = \pm\frac{5}{24}$

$\sin\alpha + \sin\beta = \frac{1}{4}$

Squaring both sides, we get

$\Rightarrow (\sin\alpha + \sin\beta)^2 = \left(\frac{1}{4}\right)^2$

$\Rightarrow \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta = \frac{1}{16} \ldots\ldots\ldots\ldots\ldots\ldots (1)$

$\cos\alpha + \cos\beta = \frac{1}{3}$

Squaring both sides, we get

$\Rightarrow (\cos\alpha + \cos\beta)^2 = \left(\frac{1}{3}\right)^2$

$\Rightarrow \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta = \frac{1}{9} \ldots\ldots\ldots\ldots\ldots\ldots (2)$

Adding equation (1) and (2), we get

$\sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta + \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta = \frac{1}{16} + \frac{1}{9}$

$\Rightarrow \sin^2\alpha + \cos^2\alpha + \sin^2\beta + \cos^2\beta + 2\sin\alpha\sin\beta + 2\cos\alpha\cos\beta = \frac{16+9}{(16)(9)}$

$\Rightarrow 1 + 1 + 2(\sin\alpha\sin\beta + \cos\alpha\cos\beta) = \frac{25}{144}$

We know,

sin A sin B + cos A cos B = cos (A – B)

Therefore,

$\Rightarrow 2 + 2(\cos(\alpha-\beta)) = \frac{25}{144}$

$\Rightarrow 2(\cos(\alpha-\beta)) = \frac{25}{144} - 2$

$\Rightarrow 2\cos(\alpha-\beta) = \frac{25-288}{144}$

$\Rightarrow \cos(\alpha-\beta) = -\frac{253}{288}$

$\left\{\because \cos x = 2\cos^2\frac{x}{2} - 1\right\}$

$\Rightarrow 2\cos^2\frac{(\alpha-\beta)}{2} - 1 = -\frac{253}{288}$

$\Rightarrow 2\cos^2\frac{(\alpha-\beta)}{2} = 1 - \frac{253}{288}$

$\Rightarrow 2\cos^2\frac{(\alpha-\beta)}{2} = \frac{288-253}{288}$

$\Rightarrow 2\cos^2\frac{(\alpha-\beta)}{2} = \frac{25}{288}$

$\Rightarrow \cos^2\frac{(\alpha-\beta)}{2} = \frac{25}{576}$

$\Rightarrow \cos\frac{\alpha-\beta}{2} = \pm\sqrt{\frac{25}{576}}$

$\Rightarrow \cos\frac{\alpha-\beta}{2} = \pm\frac{5}{24}$

Hence Proved

43. Question

If $\sin\alpha = \frac{4}{5}$ and $\cos\beta = \frac{5}{13}$, prove that $\cos\frac{\alpha-\beta}{2} = \frac{8}{\sqrt{65}}$.

Answer

Given: $\sin\alpha = \frac{4}{5}$ & $\cos\beta = \frac{5}{13}$

To prove: $\cos\frac{\alpha-\beta}{2} = \frac{8}{\sqrt{65}}$

Proof:

We know,

$\sin^2\alpha + \cos^2\alpha = 1$

$\Rightarrow \cos^2\alpha = 1 - \sin^2\alpha$

$\Rightarrow \cos\alpha = \sqrt{1-\sin^2\alpha}$

$\Rightarrow \cos\alpha = \sqrt{1-\left(\frac{4}{5}\right)^2}$

$\Rightarrow \cos\alpha = \sqrt{1-\frac{16}{25}}$

$\Rightarrow \cos\alpha = \sqrt{\frac{9}{25}}$

$\Rightarrow \cos\alpha = \frac{3}{5}$

Similarly,

$\sin^2\beta + \cos^2\beta = 1$

$\Rightarrow \sin^2\beta = 1 - \cos^2\beta$

$\Rightarrow \sin\beta = \sqrt{1-\cos^2\beta}$

$\Rightarrow \sin\beta = \sqrt{1-\left(\frac{5}{13}\right)^2}$

$$\Rightarrow \sin\beta = \sqrt{1-\frac{25}{169}}$$

$$\Rightarrow \sin\beta = \sqrt{\frac{144}{169}}$$

$$\Rightarrow \sin\beta = \frac{12}{13}$$

Identity used:

cos (α – β) = cos α cos β + sin α sin β

$$\Rightarrow \cos(\alpha-\beta) = \frac{3}{5}\times\frac{5}{13}+\frac{4}{5}\times\frac{12}{13}$$

$$\Rightarrow 2\cos^2\left(\frac{\alpha-\beta}{2}\right)-1 = \frac{15}{65}+\frac{48}{65}$$

$$\Rightarrow 2\cos^2\left(\frac{\alpha-\beta}{2}\right) = \frac{63}{65}+1$$

$$\Rightarrow 2\cos^2\left(\frac{\alpha-\beta}{2}\right) = \frac{63+65}{65}$$

$$\Rightarrow 2\cos^2\left(\frac{\alpha-\beta}{2}\right) = \frac{128}{65}$$

$$\Rightarrow \cos^2\left(\frac{\alpha-\beta}{2}\right) = \frac{64}{65}$$

$$\Rightarrow \cos\left(\frac{\alpha-\beta}{2}\right) = \sqrt{\frac{64}{65}}$$

$$\Rightarrow \cos\left(\frac{\alpha-\beta}{2}\right) = \frac{8}{\sqrt{65}}$$

Hence Proved

44 A. Question

If a cos 2x + b sin 2x = c has α and β as its roots, then prove that

$$\tan\alpha+\tan\beta = \frac{2b}{a+c}$$

Answer

Given: a cos 2x + b sin 2x = c

To prove: $\tan\alpha+\tan\beta = \frac{2b}{a+c}$

We know,

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

Therefore,

a cos 2x + b sin 2x = c

$$\Rightarrow a\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)+b\left(\frac{2\tan x}{1+\tan^2 x}\right)=c$$

$$\Rightarrow \frac{a(1-\tan^2 x)}{1+\tan^2 x}+\frac{2b\tan x}{1+\tan^2 x}=c$$

$$\Rightarrow \frac{a(1-\tan^2 x)+2b\tan x}{1+\tan^2 x}=c$$

$$\Rightarrow a(1-\tan^2 x)+2b\tan x=c(1+\tan^2 x)$$

$$\Rightarrow 2b\tan x+a-a\tan^2 x=c+c\tan^2 x$$

$$\Rightarrow 2b\tan x+a-a\tan^2 x-c-c\tan^2 x=0$$

$$\Rightarrow (-a-c)\tan^2 x+2b\tan x+a-c=0$$

We know,

If m and n are roots of the equation $ax^2 + bx + c = 0$

then,

Sum of the roots(m+n) , $= -\frac{b}{a}$

Therefore,

If tan α and tan β are the roots of the equation

$$(-a-c)\tan^2 x+2b\tan x+a-c=0$$

then,

$$\tan\alpha+\tan\beta=\frac{-2b}{-a-c}$$

$$\Rightarrow \tan\alpha+\tan\beta=\frac{-2b}{-(a+c)}$$

$$\Rightarrow \tan\alpha+\tan\beta=\frac{2b}{a+c}$$

Hence Proved

44 B. Question

If a cos 2x + b sin 2x = c has α and β as its roots, then prove that

$$\tan\alpha\tan\beta=\frac{c-a}{c+a}$$

Answer

Given: a cos 2x + b sin 2x = c

To prove: $\tan\alpha\tan\beta=\frac{c-a}{c+a}$

We know,

$$\sin 2x=\frac{2\tan x}{1+\tan^2 x}$$

$$\cos 2x=\frac{1-\tan^2 x}{1+\tan^2 x}$$

Therefore,

a cos 2x + b sin 2x = c

$$\Rightarrow a\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)+b\left(\frac{2\tan x}{1+\tan^2 x}\right)=c$$

$$\Rightarrow \frac{a(1-\tan^2 x)}{1+\tan^2 x}+\frac{2b\tan x}{1+\tan^2 x}=c$$

$$\Rightarrow \frac{a(1-\tan^2 x)+2b\tan x}{1+\tan^2 x}=c$$

$$\Rightarrow a(1-\tan^2 x)+2b\tan x=c(1+\tan^2 x)$$

$$\Rightarrow 2b\tan x+a-a\tan^2 x=c+c\tan^2 x$$

$$\Rightarrow 2b\tan x+a-a\tan^2 x-c-c\tan^2 x=0$$

$$\Rightarrow (-a-c)\tan^2 x+2b\tan x+a-c=0$$

We know,

If m and n are roots of the equation $ax^2 + bx + c = 0$

then,

Product of the roots(mn), $= \frac{c}{a}$

Therefore,

If tan α and tan β are the roots of the equation

$$(-a-c)\tan^2 x+2b\tan x+a-c=0$$

then,

$$\tan\alpha\tan\beta=\frac{a-c}{-a-c}$$

$$\Rightarrow \tan\alpha\tan\beta=\frac{-(c-a)}{-(c+a)}$$

$$\Rightarrow \tan\alpha\ \tan\beta=\frac{c-a}{c+a}$$

Hence Proved

44 C. Question

If a cos 2x + b sin 2x = c has α and β as its roots, then prove that

$$\tan(\alpha+\beta)=\frac{b}{a}$$

Answer

To prove: $\tan(\alpha+\beta)=\frac{b}{a}$

We know,

$$\tan(x+y)=\frac{\tan x+\tan y}{1+\tan x\tan y}$$

Therefore,

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1+\tan\alpha\tan\beta}$$

From previous question:

$$\tan\alpha+\tan\beta=\frac{2b}{a+c}\ \&\ \tan\alpha\ \tan\beta=\frac{c-a}{c+a}$$

$$\Rightarrow \tan(\alpha+\beta)=\frac{\frac{2b}{a+c}}{1+\frac{c-a}{c+a}}$$

$$\Rightarrow \tan(\alpha+\beta)=\frac{\frac{2b}{a+c}}{\frac{c+a+c-a}{c+a}}$$

$$\Rightarrow \tan(\alpha+\beta)=\frac{2b}{2c}$$

$$\Rightarrow \tan(\alpha+\beta)=\frac{b}{c}$$

Hence Proved

45. Question

If cos α + cos β = 0 = sin α + sin β, then prove that cos 2α + cos 2β = - 2 cos (α + β).

Answer

Given: $\cos\alpha+\cos\beta=\sin\alpha+\sin\beta=0$

To prove: $\cos 2\alpha+\ \cos 2\beta=\ -2\cos(\alpha+\ \beta)$

Proof:

cos α + cos β = 0

Squaring both sides:

$\Rightarrow (\cos\alpha+\cos\beta)^2=(0)^2$

$\Rightarrow \cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta=0$ ......(1)

sin α + sin β = 0

Squaring both sides:

$\Rightarrow (\sin\alpha+\sin\beta)^2=(0)^2$

$\Rightarrow \sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta=0$ ....(2)

Subtracting equation (1) from (2), we get

$\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta-(\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta)=0$

$\Rightarrow \cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta-\sin^2\alpha-\sin^2\beta-2\sin\alpha\sin\beta=0$

$\Rightarrow \cos^2\alpha-\sin^2\alpha+\cos^2\beta-\sin^2\beta+2(\cos\alpha\cos\beta-\sin\alpha\sin\beta)=0$

$\{\because \cos^2 x-\sin^2 x=2x\ \&$

$\cos A\cos B-\sin A\sin B=\cos(A+B)\}$

$\Rightarrow \cos 2\alpha+\cos 2\beta+2\cos(\alpha+\beta)=0$

$\Rightarrow \cos 2\alpha+\cos 2\beta=-2\cos(\alpha+\beta)$

Hence Proved

Exercise 9.2

1. Question

Prove that:

$\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$

Answer

LHS is

$\sin 5x = \sin(3x+2x)$

But we know,

$\sin(x+y) = \sin x \cos y+\cos x \sin y$.....(i)

$\Rightarrow \sin 5x = \sin 3x \cos 2x+\cos 3x \sin 2x$

$\Rightarrow \sin 5x = \sin (2x+x) \cos 2x+\cos (2x+x) \sin 2x$........(ii)

And

$\cos (x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$......(iii)

Now substituting equation (i) and (iii) in equation (ii), we get

$\Rightarrow \sin 5x = (\sin 2x \cos x+\cos 2x \sin x)\cos 2x+(\cos 2x \cos x - \sin 2x \sin x) \sin 2x$

$\Rightarrow \sin 5x = \sin 2x \cos 2x \cos x+\cos^2 2x \sin x+(\sin 2x \cos 2x \cos x - \sin^2 2x \sin x)$

$\Rightarrow \sin 5x = 2\sin 2x \cos 2x \cos x+\cos^2 2x \sin x- \sin^2 2x \sin x$ ....(iv)

Now $\sin 2x = 2\sin x \cos x$.........(v)

And $\cos 2x = \cos^2 x -\sin^2 x$.........(vi)

Substituting equation (v) and (vi) in equation (iv), we get

$\Rightarrow \sin 5x = 2(2\sin x \cos x)(\cos^2 x -\sin^2 x)\cos x+(\cos^2 x -\sin^2 x)^2 \sin x -(2\sin x \cos x)^2 \sin x$

$\Rightarrow \sin 5x = 4(\sin x \cos^2 x)([1-\sin^2 x] -\sin^2 x)+([1-\sin^2 x]-\sin^2 x)^2 \sin x -(4\sin^2 x \cos^2 x)\sin x$ (as $\cos^2 x+\sin^2 x=1 \Rightarrow \cos^2 x=1-\sin^2 x$)

$\Rightarrow \sin 5x = 4(\sin x [1-\sin^2 x])(1-2\sin^2 x)+(1-2\sin^2 x)^2 \sin x-4\sin^3 x [1-\sin^2 x]$

$\Rightarrow \sin 5x = 4\sin x(1-\sin^2 x)(1-2\sin^2 x)+(1-4\sin^2 x+4\sin^4 x)\sin x-4\sin^3 x +4\sin^5 x$

$\Rightarrow \sin 5x = (4\sin x-4\sin^3 x)(1-2\sin^2 x) +\sin x-4\sin^3 x+4\sin^5 x-4\sin^3 x +4\sin^5 x$

$\Rightarrow \sin 5x = 4\sin x-8\sin^3 x-4\sin^3 x+8\sin^5 x+\sin x-8\sin^3 x+8\sin^5 x$

$\Rightarrow \sin 5x = 5\sin x-20\sin^3 x+16\sin^5 x$

Hence LHS = RHS

[Hence proved]

2. Question

Prove that:

$4(\cos^3 10^o + \sin^3 20^o) = 3 (\cos 10^o + \sin 20^o)$

Answer

We know that

$\sin 60° = \frac{\sqrt{3}}{2} = \cos 30°$ $\Rightarrow \sin (3\times 20°)=\cos (3\times 10°)$

$\Rightarrow 3\sin 20°-4\sin^3 20°=4\cos^3 10°-3\cos 10°$

(as $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$)

$\Rightarrow 4(\cos^3 10° + \sin^3 20°) = 3(\sin 20° + \cos 10°)$

LHS=RHS

Hence proved

3. Question

Prove that:

$$\cos^3 x \sin 3x + \sin^3 x \cos 3x = \frac{3}{4}\sin 4x$$

Answer

We know that,

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow 4\cos^3\theta = \cos 3\theta + 3\cos\theta$

$$\Rightarrow \cos^3\theta = \frac{\cos 3\theta + 3\cos\theta}{4} \dots \text{(i)}$$

And similarly

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow 4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$$\Rightarrow \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4} \dots \text{(ii)}$$

Now,

$\text{LHS} = \cos^3 x \sin 3x + \sin^3 x \cos 3x$

Substituting the values from equation (i) and (ii), we get

$$\Rightarrow = \left(\frac{\cos 3x + 3\cos x}{4}\right)\sin 3x + \left(\frac{3\sin x - \sin 3x}{4}\right)\cos 3x$$

$$\Rightarrow = \frac{1}{4}(\sin 3x \cos 3x + 3\sin 3x \cos x + 3\sin x \cos 3x - \sin 3x \cos 3x)$$

$$\Rightarrow = \frac{1}{4}(3[\sin 3x \cos x + \sin x \cos 3x] + 0)$$

$$\Rightarrow \frac{1}{4}(3\sin(3x + x))$$

(as $\sin(x+y) = \sin x \cos y + \cos x \sin y$)

$$\Rightarrow \frac{3}{4}\sin 4x$$

=RHS

Hence Proved

4. Question

Prove that:

$$\tan x \tan\left(x+\frac{\pi}{3}\right)+\tan x \tan\left(\frac{\pi}{3}-x\right)+\tan\left(x+\frac{\pi}{3}\right)\tan\left(x-\frac{\pi}{3}\right)=-3$$

Answer

$$\text{LHS} = \tan x \tan\left(x+\frac{\pi}{3}\right)+\tan x \tan\left(\frac{\pi}{3}-x\right)+\tan\left(x+\frac{\pi}{3}\right)\tan\left(x-\frac{\pi}{3}\right)$$

$$\Rightarrow = \tan x\left(\frac{\tan x+\tan\frac{\pi}{3}}{1-\tan x\tan\frac{\pi}{3}}\right)+\tan x\left(\frac{\tan\frac{\pi}{3}-\tan x}{1+\tan x\tan\frac{\pi}{3}}\right)$$
$$+\left(\frac{\tan x+\tan\frac{\pi}{3}}{1-\tan x\tan\frac{\pi}{3}}\right)\left(\frac{\tan\frac{\pi}{3}-\tan x}{1+\tan x\tan\frac{\pi}{3}}\right)$$

$$\left(\because \tan(A+B)=\left(\frac{\tan A+\tan B}{1-\tan A\tan B}\right) \text{ and } \tan(A-B)=\left(\frac{\tan A-\tan B}{1+\tan A\tan B}\right)\right)$$

$$\Rightarrow = \tan x\left(\frac{\tan x+\sqrt{3}}{1-\tan x(\sqrt{3})}\right)+\tan x\left(\frac{\sqrt{3}-\tan x}{1+\tan x(\sqrt{3})}\right)$$
$$+\left(\frac{\tan x+\sqrt{3}}{1-\tan x(\sqrt{3})}\right)\left(\frac{\sqrt{3}-\tan x}{1+\tan x(\sqrt{3})}\right)$$

$$\left(\text{as} \tan\frac{\pi}{3}=\sqrt{3}\right)$$

$$=\left(\frac{\left(1+\tan x(\sqrt{3})\right)\tan x(\tan x+\sqrt{3})+\left(1-\tan x(\sqrt{3})\right)\tan x(\sqrt{3}-\tan x)+(\tan x+\sqrt{3})(\sqrt{3}-\tan x)}{\left(1-\tan x(\sqrt{3})\right)\left(1+\tan x(\sqrt{3})\right)}\right)$$

$$=\left(\frac{(1+\sqrt{3}\tan x)\tan x(\tan x+\sqrt{3})+(1-\sqrt{3}\tan x)\tan x(\sqrt{3}-\tan x)+\left(\tan^2 x-(\sqrt{3})^2\right)}{\left(1-(\sqrt{3}\tan x)^2\right)}\right)$$

$$=\left(\frac{(\tan x+\sqrt{3}\tan^2 x)(\tan x+\sqrt{3})+(\tan x-\sqrt{3}\tan^2 x)(\sqrt{3}-\tan x)+(\tan^2 x-3)}{\left(1-(\sqrt{3}\tan x)^2\right)}\right)$$

$$=\left(\frac{(\cancel{\tan^2 x}+\sqrt{3}\tan x+\sqrt{3}\tan^3 x+\cancel{3\tan^2 x})+(\sqrt{3}\tan x-\cancel{3\tan^2 x}-\cancel{\tan^2 x}+\sqrt{3}\tan^3 x)+(\tan^2 x-3)}{\left(1-(\sqrt{3}\tan x)^2\right)}\right)$$

$$=\left(\frac{(0+2\sqrt{3}\tan x+2\sqrt{3}\tan^3 x+3\tan^2 x)+(\tan^2 x-3)}{\left(1-(\sqrt{3}\tan x)^2\right)}\right)$$

$$=\left(\frac{2\sqrt{3}\tan x+2\sqrt{3}\tan^3 x+4\tan^2 x-3}{(1-3\tan^2 x)}\right)$$

$\neq -3$

Hence LHS$\neq$ RHS

5. Question

Prove that:

$$\tan x+\tan\left(\frac{\pi}{3}+x\right)-\tan\left(\frac{\pi}{3}-x\right)=3\tan 3x$$

Answer

$$\text{LHS} = \tan x + \tan\left(\frac{\pi}{3} + x\right) - \tan\left(\frac{\pi}{3} - x\right)$$

$$\Rightarrow = \tan x + \left(\frac{\tan\frac{\pi}{3} + \tan x}{1 - \tan x \tan\frac{\pi}{3}}\right) - \left(\frac{\tan\frac{\pi}{3} - \tan x}{1 + \tan x \tan\frac{\pi}{3}}\right)$$

$$\left(\because \tan(A+B) = \left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) \text{ and } \tan(A-B) = \left(\frac{\tan A - \tan B}{1 + \tan A \tan B}\right)\right)$$

$$\Rightarrow = \tan x + \left(\frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}\right) - \left(\frac{\sqrt{3} - \tan x}{1 + \sqrt{3}\tan x}\right)$$

$$\Rightarrow = \tan x + \left(\frac{(1 + \sqrt{3}\tan x)(\sqrt{3} + \tan x) - (1 - \sqrt{3}\tan x)(\sqrt{3} - \tan x)}{\left(1 - \tan x\left(\sqrt{3}\right)\right)\left(1 + \tan x\left(\sqrt{3}\right)\right)}\right)$$

$$\Rightarrow$$
$$= \tan x$$
$$+ \left(\frac{(\cancel{\sqrt{3}} + 3\tan x + \tan x + \cancel{\sqrt{3}\tan^2 x}) - (\cancel{\sqrt{3}} - 3\tan x - \tan x + \cancel{\sqrt{3}\tan^2 x})}{(1 - 3\tan^2 x)}\right)$$

$$\Rightarrow = \tan x + \left(\frac{(0 + 6\tan x + 2\tan x + 0)}{(1 - 3\tan^2 x)}\right)$$

$$\Rightarrow = \tan x + \left(\frac{8\tan x}{(1 - 3\tan^2 x)}\right)$$

$$\Rightarrow = \left(\frac{\tan x(1 - 3\tan^2 x) + 8\tan x}{(1 - 3\tan^2 x)}\right)$$

$$\Rightarrow = \left(\frac{(\tan x - 3\tan^3 x) + 8\tan x}{(1 - 3\tan^2 x)}\right)$$

$$\Rightarrow = \left(\frac{9\tan x - 3\tan^3 x}{(1 - 3\tan^2 x)}\right)$$

$$\Rightarrow = 3\left(\frac{3\tan x - \tan^3 x}{(1 - 3\tan^2 x)}\right)$$

$$\Rightarrow = 3\tan 3x = \text{RHS}$$

$$\left(\text{as } \tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\right)$$

Hence proved

6. Question

Prove that:

$$\cot x + \cot\left(\frac{\pi}{3} + x\right) - \cot\left(\frac{\pi}{3} - x\right) = 3\cot 3x$$

Answer

$$\text{LHS} = \cot x + \cot\left(\frac{\pi}{3} + x\right) - \cot\left(\frac{\pi}{3} - x\right)$$

$$\Rightarrow \frac{1}{\tan x}+\frac{1}{\tan\left(\frac{\pi}{3}+x\right)}-\frac{1}{\tan\left(\frac{\pi}{3}-x\right)}$$

$$\Rightarrow=\frac{1}{\tan x}+\left(\frac{1-\tan x\tan\frac{\pi}{3}}{\tan\frac{\pi}{3}+\tan x}\right)-\left(\frac{1+\tan x\tan\frac{\pi}{3}}{\tan\frac{\pi}{3}-\tan x}\right)$$

$$\left(\because \tan(A+B)=\left(\frac{\tan A+\tan B}{1-\tan A\tan B}\right) \text{ and } \tan(A-B)=\left(\frac{\tan A-\tan B}{1+\tan A\tan B}\right)\right)$$

$$\Rightarrow=\frac{1}{\tan x}+\left(\frac{1-\sqrt{3}\tan x}{\sqrt{3}+\tan x}\right)-\left(\frac{1+\sqrt{3}\tan x}{\sqrt{3}-\tan x}\right)$$

$$\Rightarrow=\frac{1}{\tan x}+\left(\frac{(1-\sqrt{3}\tan x)(\sqrt{3}-\tan x)-(1+\sqrt{3}\tan x)(\sqrt{3}+\tan x)}{(\sqrt{3}+\tan x)(\sqrt{3}-\tan x)}\right)$$

$$\Rightarrow =\frac{1}{\tan x}+\left(\frac{(\sqrt{3}-\tan x-3\tan x+\sqrt{3}\tan^2 x)-(\sqrt{3}+3\tan x+\tan x+\sqrt{3}\tan^2 x)}{(3-\tan^2 x)}\right)$$

$$\Rightarrow=\frac{1}{\tan x}+\left(\frac{(0-4\tan x-4\tan x+0)}{(3-\tan^2 x)}\right)$$

$$\Rightarrow=\frac{1}{\tan x}-\left(\frac{8\tan x}{((3-\tan^2 x))}\right)$$

$$\Rightarrow=\left(\frac{(3-\tan^2 x)-8\tan^2 x}{\tan x\,(3-\tan^2 x)}\right)$$

$$\Rightarrow=\left(\frac{3-9\tan^2 x}{(3\tan x-\tan^3 x)}\right)$$

$$\Rightarrow=3\left(\frac{1-3\tan^2 x}{(3\tan x-\tan^3 x)}\right)$$

$$\Rightarrow=3\times\frac{1}{\tan 3x}$$

$$\left(\text{as } \tan 3x=\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}\right)$$

$$\Rightarrow 3\cot 3x=\text{RHS}$$

Hence proved

7. Question

Prove that:

$$\cot x+\cot\left(\frac{\pi}{3}+x\right)+\cot\left(\frac{2\pi}{3}+x\right)=3\cot 3x$$

Answer

$$\text{LHS}=\cot x+\cot\left(\frac{\pi}{3}+x\right)+\cot\left(\frac{2\pi}{3}+x\right)$$

We know,

$\cot\left(\frac{2\pi}{3}+x\right)=\cot\left(\pi-\left(\frac{\pi}{3}-x\right)\right)=-\cot\left(\frac{\pi}{3}-x\right)$ (as –cot θ=cot (180°–θ))

Hence the above LHS becomes

$$=\cot x+\cot\left(\frac{\pi}{3}+x\right)-\cot\left(\frac{\pi}{3}-x\right)$$

$$\Rightarrow \frac{1}{\tan x}+\frac{1}{\tan\left(\frac{\pi}{3}+x\right)}-\frac{1}{\tan\left(\frac{\pi}{3}-x\right)}$$

$$\Rightarrow=\frac{1}{\tan x}+\left(\frac{1-\tan x\tan\frac{\pi}{3}}{\tan\frac{\pi}{3}+\tan x}\right)-\left(\frac{1+\tan x\tan\frac{\pi}{3}}{\tan\frac{\pi}{3}-\tan x}\right)$$

$$\left(\because \tan(A+B)=\left(\frac{\tan A+\tan B}{1-\tan A\tan B}\right)\text{ and }\tan(A-B)=\left(\frac{\tan A-\tan B}{1+\tan A\tan B}\right)\right)$$

$$\Rightarrow=\frac{1}{\tan x}+\left(\frac{1-\sqrt{3}\tan x}{\sqrt{3}+\tan x}\right)-\left(\frac{1+\sqrt{3}\tan x}{\sqrt{3}-\tan x}\right)$$

$$\Rightarrow=\frac{1}{\tan x}+\left(\frac{(1-\sqrt{3}\tan x)(\sqrt{3}-\tan x)-(1+\sqrt{3}\tan x)(\sqrt{3}+\tan x)}{(\sqrt{3}+\tan x)(\sqrt{3}-\tan x)}\right)$$

$\Rightarrow$

$$=\frac{1}{\tan x}+\left(\frac{(\cancel{\sqrt{3}}-\tan x-3\tan x+\cancel{\sqrt{3}\tan^2 x})-(\cancel{\sqrt{3}}+3\tan x+\tan x+\cancel{\sqrt{3}\tan^2 x})}{(3-\tan^2 x)}\right)$$

$$\Rightarrow=\frac{1}{\tan x}+\left(\frac{(0-4\tan x-4\tan x+0)}{(3-\tan^2 x)}\right)$$

$$\Rightarrow=\frac{1}{\tan x}-\left(\frac{8\tan x}{((3-\tan^2 x))}\right)$$

$$\Rightarrow=\left(\frac{(3-\tan^2 x)-8\tan^2 x}{\tan x\,(3-\tan^2 x)}\right)$$

$$\Rightarrow=\left(\frac{3-9\tan^2 x}{(3\tan x-\tan^3 x)}\right)$$

$$\Rightarrow=3\left(\frac{1-3\tan^2 x}{(3\tan x-\tan^3 x)}\right)$$

$$\Rightarrow=3\times\frac{1}{\tan 3x}$$

$$\left(\text{as }\tan 3x=\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}\right)$$

$$\Rightarrow 3\cot 3x=\text{RHS}$$

Hence proved

8. Question

Prove that:

$\sin 5x = 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x$

Answer

LHS is

$\sin 5x = \sin(3x+2x)$

But we know,

$\sin(x+y) = \sin x \cos y+\cos x \sin y$.....(i)

$\Rightarrow \sin 5x = \sin 3x \cos 2x+\cos 3x \sin 2x$

$\Rightarrow \sin 5x = \sin (2x+x) \cos 2x+\cos (2x+x) \sin 2x$........(ii)

And

$\cos (x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$......(iii)

Now substituting equation (i) and (iii) in equation (ii), we get

$\Rightarrow \sin 5x = (\sin 2x \cos x+\cos 2x \sin x)(\cos 2x)+(\cos 2x \cos x - \sin 2x \sin x)(\sin 2x)$........(iv)

Now $\sin 2x = 2\sin x \cos x$.........(v)

And $\cos 2x = \cos^2 x - \sin^2 x$.........(vi)

Substituting equation (v) and (vi) in equation (iv), we get

$\Rightarrow \sin 5x =[(2 \sin x \cos x)\cos x+(\cos^2 x-\sin^2 x)\sin x](\cos^2 x-\sin^2 x)+[(\cos^2 x-\sin^2 x)\cos x - (2 \sin x \cos x) \sin x](2 \sin x \cos x)$

$\Rightarrow \sin 5x =[2 \sin x \cos^2 x+\sin x\cos^2 x-\sin^3 x](\cos^2 x-\sin^2 x)+[\cos^3 x-\sin^2 x\cos x - 2 \sin^2 x \cos x](2 \sin x \cos x)$

$\Rightarrow \sin 5x =\cos^2 x[3 \sin x \cos^2 x -\sin^3 x]-\sin^2 x[3 \sin x \cos^2 x-\sin^3 x]+2 \sin x \cos^4 x-2 \sin^3 x \cos^2 x - 4 \sin^3 x \cos^2 x$

$\Rightarrow \sin 5x = 3 \sin x \cos^4 x -\sin^3 x\cos^2 x- 3 \sin^3 x \cos^2 x-\sin^5 x +2 \sin x \cos^4 x-2 \sin^3 x \cos^2 x - 4 \sin^3 x \cos^2 x$

$\Rightarrow \sin 5x = 5 \sin x \cos^4 x -10\sin^3 x\cos^2 x +\sin^5 x$

Hence LHS = RHS

[Hence proved]

9. Question

Prove that:

$$\sin^3 x + \sin^3\left(\frac{2\pi}{3}+x\right)+\sin^3\left(\frac{4\pi}{3}+x\right) = -\frac{3}{4}\sin 3x$$

Answer

$\sin 3\theta=3\sin\theta-4\sin^3\theta$

$\Rightarrow 4\sin^3\theta=3\sin\theta-\sin 3\theta$

$$\Rightarrow \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4} \ldots (i)$$

Now,

$$\text{LHS} = \sin^3 x + \sin^3\left(\frac{2\pi}{3}+x\right)+\sin^3\left(\frac{4\pi}{3}+x\right)$$

Substituting equation (i) in above LHS, we get

$$= \frac{3\sin x - \sin 3x}{4} + \frac{3\sin\left(\frac{2\pi}{3} + x\right) - \sin 3\left(\frac{2\pi}{3} + x\right)}{4} + \frac{3\sin\left(\frac{4\pi}{3} + x\right) - \sin 3\left(\frac{4\pi}{3} + x\right)}{4} \ldots. (ii)$$

We know,

$\sin\left(\frac{2\pi}{3} + x\right) = \sin\left(\pi - \left(\frac{\pi}{3} - x\right)\right) = \sin\left(\frac{\pi}{3} - x\right) \ldots \ldots (iii)$ (as sin θ =sin (180°–θ))

Similarly,

$\sin\left(\frac{4\pi}{3} + x\right) = \sin\left(\pi + \left(\frac{\pi}{3} - x\right)\right) = -\sin\left(\frac{\pi}{3} - x\right) \ldots \ldots (iv)$ (as –sin θ =sin (180°+θ))

Substituting the equation (iii) and (iv) in equation (ii), we get

$$= \frac{3\sin x - \sin 3x}{4} + \frac{3\sin\left\{\pi - \left(\frac{\pi}{3} - x\right)\right\} - \sin(2\pi + 3x)}{4} + \frac{3\sin\left\{\pi + \left(\frac{\pi}{3} + x\right)\right\} - \sin(4\pi + 3x)}{4}$$

$$= \frac{1}{4}\left[3\sin x - \sin 3x + 3\sin\left\{\pi - \left(\frac{\pi}{3} - x\right)\right\} - \sin(2\pi + 3x) + 3\sin\left\{\pi + \left(\frac{\pi}{3} + x\right)\right\} - \sin(4\pi + 3x)\right]$$

$$= \frac{1}{4}\left[3\sin x - \sin 3x + 3\sin\left(\frac{\pi}{3} - x\right) - \sin(3x) - 3\sin\left(\frac{\pi}{3} + x\right) - \sin(3x)\right]$$

$$= \frac{1}{4}\left[3\sin x - 3\sin 3x + 3\left\{\sin\left(\frac{\pi}{3} - x\right) - 3\sin\left(\frac{\pi}{3} + x\right)\right\}\right]$$

$$= \frac{1}{4}\left[3\sin x - 3\sin 3x + 3\left\{\sin\left(\frac{\pi}{3} - x\right) - 3\sin\left(\frac{\pi}{3} + x\right)\right\}\right]$$

We know,

$$\left[\because \sin C - \sin D = 2\cos\frac{C + D}{2}\sin\frac{C - D}{2}\right]$$

Substituting this in the above equation, we get

$$= \frac{1}{4}\left[3\sin x - 3\sin 3x + 3\left\{2\cos\left(\frac{\frac{\pi}{3} - x + \frac{\pi}{3} + x}{2}\right)\sin\left(\frac{\frac{\pi}{3} - x - \frac{\pi}{3} - x}{2}\right)\right\}\right]$$

$$= \frac{3}{4}\left[\sin x - \sin 3x + 2\left\{\cos\left(\frac{\pi}{3}\right)\sin(-x)\right\}\right]$$

$$= \frac{3}{4}\left[\sin x - \sin 3x - 2\left\{\frac{1}{2}\sin x\right\}\right]$$

$$= -\frac{3}{4}\sin 3x = \text{RHS}$$

Hence proved

10. Question

Prove that:

$$\left|\sin x \sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right)\right| \le \frac{1}{4}$$ For all values of x.

Answer

We know

sin (A+B)sin (A–B)=$\sin^2$A–$\sin^2$B

So the above LHS becomes,

$$\left|\sin x \sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right)\right|$$

$$\Rightarrow \left|\sin x\left\{\sin^2\frac{\pi}{3}-\sin^2 x\right\}\right|$$

$$\Rightarrow \left|\sin x\left\{\left(\frac{\sqrt{3}}{2}\right)^2-\sin^2 x\right\}\right|$$

$$\Rightarrow \left|\sin x\left\{\frac{3}{4}-\sin^2 x\right\}\right|$$

$$\Rightarrow \frac{1}{4}|3\sin x - 4\sin^3 x|$$

But 3sin x–4 $\sin^3$x=sin 3x

$$\Rightarrow \frac{1}{4}|\sin 3x|$$

But |sin θ|≤ 1 for all values of x

Hence LHS $\le \frac{1}{4}$

Therefore $\left|\sin x \sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right)\right| \le \frac{1}{4}$ For all values of x

11. Question

Prove that:

$$\left|\cos x \cos\left(\frac{\pi}{3}-x\right)\cos\left(\frac{\pi}{3}+x\right)\right| \le \frac{1}{4}$$ for all values of x

Answer

We know

cos (A+B)cos (A–B)=$\cos^2$A–$\sin^2$B

So the above LHS becomes,

$$\left|\cos x \cos\left(\frac{\pi}{3}-x\right)\cos\left(\frac{\pi}{3}+x\right)\right|$$

$$\Rightarrow \left|\cos x\left\{\cos^2\frac{\pi}{3}-\sin^2 x\right\}\right|$$

$$\Rightarrow \left|\cos x\left\{\left(\frac{1}{2}\right)^2-\sin^2 x\right\}\right|$$

$$\Rightarrow \left|\cos x\left\{\frac{1}{4}-(1-\cos^2 x)\right\}\right|$$

$$\Rightarrow \frac{1}{4}|\cos x - 4\cos x + 4\cos^3 x|$$

$$\Rightarrow \frac{1}{4}|4\cos^3 x - 3\cos x|$$

But $4\cos^3 x - 3\cos x = \cos 3x$

$$\Rightarrow \frac{1}{4}|\cos 3x|$$

But $|\cos \theta| \le 1$ for all values of x

Hence LHS $\le \frac{1}{4}$

Therefore $\left|\cos x \cos\left(\frac{\pi}{3} - x\right)\cos\left(\frac{\pi}{3} + x\right)\right| \le \frac{1}{4}$ For all values of x

Exercise 9.3

1. Question

Prove that:

$$\sin^2\frac{2\pi}{5} - \sin^2\frac{\pi}{3} = \frac{\sqrt{5}-1}{8}$$

Answer

$$\text{LHS} = \sin^2\frac{2\pi}{5} - \sin^2\frac{\pi}{3}$$

$$= \sin^2\left(\frac{\pi}{2} - \frac{\pi}{10}\right) - \sin^2\frac{\pi}{3}$$

But $\sin(90° - \theta) = \cos\theta$

Then the above equation becomes,

$$= \cos^2\left(\frac{\pi}{10}\right) - \left(\frac{\sqrt{3}}{2}\right)^2$$

And $\because \cos\frac{\pi}{10} = \frac{\sqrt{10+2\sqrt{5}}}{4}$

Hence the above equation becomes,

$$= \left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 - \frac{3}{4}$$

$$= \frac{10+2\sqrt{5}}{16} - \frac{3}{4}$$

$$= \frac{10+2\sqrt{5}-12}{16}$$

$$= \frac{2\sqrt{5}-2}{16}$$

$$= \frac{\sqrt{5}-1}{8} = \text{RHS}$$

Hence proved

2. Question

Prove that:

$$\sin^2 24° - \sin^2 6° = \frac{\sqrt{5}-1}{8}$$

Answer

$LHS = \sin^2 24° - \sin^2 6°$

But sin (A+B)sin(A–B)=$\sin^2$A–$\sin^2$B

Then the above equation becomes,

$= \sin(24° + 6°) - \sin(24° - 6°)$

$= \sin(30°) - \sin(18°)$

And $\because \sin(18°) = \frac{\sqrt{5}-1}{4}$

Hence the above equation becomes,

$= \frac{1}{2} \times \frac{\sqrt{5}-1}{4}$

$\frac{\sqrt{5}-1}{8} = RHS$

Hence proved

3. Question

Prove that:

$$\sin^2 42° - \cos^2 78° = \frac{\sqrt{5}+1}{8}$$

Answer

$LHS = \sin^2 42° - \cos^2 78°$

$\Rightarrow = \sin^2(90° - 48°) - \cos^2(90° - 12°)$

$= \cos^2 48° - \sin^2 12°$ $(\because \sin(90 - \theta) = \cos\theta$ and $\cos(90 - \theta) = \sin\theta)$

But cos (A+B)cos(A–B)=$\cos^2$A–$\sin^2$B

Then the above equation becomes,

$= \cos(48° + 12°)\cos(48° - 12°)$

$= \cos(60°)\cos(36°)$

And $\because \cos(36°) = \frac{\sqrt{5}+1}{4}$

Hence the above equation becomes,

$= \frac{1}{2} \times \frac{\sqrt{5}+1}{4}$

$= \frac{\sqrt{5}+1}{8} = RHS$

Hence proved

4. Question

Prove that:

$$\cos 78^\circ \cos 42^\circ \cos 36^\circ = \frac{1}{8}$$

Answer

$\text{LHS} = \cos 78^\circ \cos 42^\circ \cos 36^\circ$

Multiply and divide by 2, we get

$$= \frac{1}{2}(2\cos 78^\circ \cos 42^\circ \cos 36^\circ)$$

But 2cos A cos B = cos(A+B)+cos(A–B)

Then the above equation becomes,

$$= \frac{1}{2}(\cos(78^\circ + 42^\circ) + \cos(78^\circ - 42^\circ)) \times \cos 36^\circ$$

$$= \frac{1}{2}(\cos 120^\circ + \cos 36^\circ)\cos 36^\circ$$

$$= \frac{1}{2}(\cos(180^\circ - 60^\circ) + \cos 36^\circ)\cos 36^\circ$$

But cos(180°–θ)=–cos θ

So the above equation becomes,

$$= \frac{1}{2}(-\cos(60^\circ) + \cos 36^\circ)\cos 36^\circ$$

And $\because \cos(36^\circ) = \frac{\sqrt{5}+1}{4}$

Hence the above equation becomes,

$$= \frac{1}{2}\left(-\frac{1}{2} + \frac{\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1-2}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{(\sqrt{5})^2 - 1^2}{16}\right)$$

$$= \frac{1}{2}\left(\frac{5-1}{16}\right)$$

$$= \frac{1}{2}\left(\frac{4}{16}\right)$$

$$= \frac{1}{8} = \text{RHS}$$

Hence proved

5. Question

Prove that:

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{16}$$

Answer

$$\text{LHS}=\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}$$

Multiply and divide by $2\sin\frac{\pi}{15}$, we get

$$=\frac{\left(2\sin\frac{\pi}{15}\cos\frac{\pi}{15}\right)\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}}{2\sin\frac{\pi}{15}}$$

But 2sin A cos A = sin 2A

Then the above equation becomes,

$$=\frac{\left(\sin\frac{2\pi}{15}\right)\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}}{2\sin\frac{\pi}{15}}$$

Multiply and divide by 2, we get

$$=\frac{\left(2\sin\frac{2\pi}{15}\cos\frac{2\pi}{15}\right)\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}}{2\times 2\sin\frac{\pi}{15}}$$

But 2sin A cos A = sin 2A

Then the above equation becomes,

$$=\frac{\left(\sin\frac{4\pi}{15}\right)\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}}{4\sin\frac{\pi}{15}}$$

Multiply and divide by 2, we get

$$=\frac{\left(2\sin\frac{4\pi}{15}\cos\frac{4\pi}{15}\right)\cos\frac{7\pi}{15}}{2\times 4\sin\frac{\pi}{15}}$$

But 2sin A cos A = sin 2A

Then the above equation becomes,

$$=\frac{\left(\sin\frac{8\pi}{15}\right)\cos\frac{7\pi}{15}}{8\sin\frac{\pi}{15}}$$

Multiply and divide by 2, we get

$$=\frac{\left(2\sin\frac{8\pi}{15}\cos\frac{7\pi}{15}\right)}{2\times 8\sin\frac{\pi}{15}}$$

But 2sin A cos B = sin (A+B) +sin(A–B), so the above equation becomes,

$$= \frac{\sin\left(\frac{8\pi}{15} + \frac{7\pi}{15}\right) + \sin\left(\frac{8\pi}{15} - \frac{7\pi}{15}\right)}{16 \sin\frac{\pi}{15}}$$

$$= \frac{\sin(\pi) + \sin\left(\frac{\pi}{15}\right)}{16 \sin\frac{\pi}{15}}$$

$$= \frac{0 + \sin\left(\frac{\pi}{15}\right)}{16 \sin\frac{\pi}{15}}$$

$$= \frac{\sin\left(\frac{\pi}{15}\right)}{16 \sin\frac{\pi}{15}}$$

$$= \frac{1}{16} = \text{RHS}$$

Hence proved

6. Question

Prove that:

$$\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ = \frac{1}{16}$$

Answer

$\text{LHS} = \cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ$

By regrouping the LHS and multiplying and dividing by 4 we get,

$$= \frac{1}{4}(2\cos 66^\circ \cos 6^\circ)(2\cos 78^\circ \cos 42^\circ)$$

But 2cos A cos B = cos (A+B) +cos (A–B)

Then the above equation becomes,

$$= \frac{1}{4}(\cos(66^\circ + 6^\circ) + \cos(66^\circ - 6^\circ))(\cos(78^\circ + 42^\circ) + \cos(78^\circ - 42^\circ))$$

$$= \frac{1}{4}(\cos(72^\circ) + \cos(60^\circ))(\cos(120^\circ) + \cos(36^\circ))$$

$$= \frac{1}{4}(\cos(90^\circ - 18^\circ) + \cos(60^\circ))(\cos(180^\circ - 60^\circ) + \cos(36^\circ))$$

But cos(90°–θ)=sin θ and cos(180°–θ)=–cos(θ).

Then the above equation becomes,

$$= \frac{1}{4}(\sin(18^\circ) + \cos(60^\circ))(-\cos(60^\circ) + \cos(36^\circ))$$

Now, $\cos(36^\circ) = \frac{\sqrt{5}+1}{4}$

$$\sin(18^\circ) = \frac{\sqrt{5} - 1}{4}$$

$$\cos(60^\circ) = \frac{1}{2}$$

Substituting the corresponding values, we get

$$=\frac{1}{4}\left(\frac{\sqrt{5}-1}{4}+\frac{1}{2}\right)\left(-\frac{1}{2}+\frac{\sqrt{5}+1}{4}\right)$$

$$=\frac{1}{4}\left(\frac{\sqrt{5}-1+2}{4}\right)\left(\frac{\sqrt{5}+1-2}{4}\right)$$

$$=\frac{1}{4}\left(\frac{\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}-1}{4}\right)$$

$$=\frac{1}{4}\left(\frac{\left(\sqrt{5}\right)^2-1^2}{4\times 4}\right)$$

$$=\frac{1}{4}\left(\frac{4}{16}\right)$$

$$=\frac{1}{16}=\text{RHS}$$

Hence proved

7. Question

Prove that:

$$\sin 6°\sin 42°\sin 66°\sin 78°=\frac{1}{16}$$

Answer

$\text{LHS}=\sin 6°\sin 42°\sin 66°\sin 78°$

By regrouping the LHS and multiplying and dividing by 4 we get,

$$=\frac{1}{4}(2\sin 66°\sin 6°)(2\sin 78°\sin 42°)$$

But 2sin A sin B = cos (A–B) –cos (A+B)

Then the above equation becomes,

$$=\frac{1}{4}(\cos(66°-6°)-\cos(66°+6°))(\cos(78°-42°)-\cos(78°+42°))$$

$$=\frac{1}{4}(\cos(60°)-\cos(72°))(\cos(36°)-\cos(120°))$$

$$=\frac{1}{4}(\cos(60°)-\cos(90°-18°))(\cos(36°)-\cos(180°-60°))$$

But cos(90°–θ)=sin θ and cos(180°–θ)=–cos(θ).

Then the above equation becomes,

$$=\frac{1}{4}(\cos(60°)-\sin(18°))(\cos(36°)+\cos(60°))$$

Now, $\cos(36°)=\frac{\sqrt{5}+1}{4}$

$$\sin(18°)=\frac{\sqrt{5}-1}{4}$$

$\cos(60°) = \frac{1}{2}$

Substituting the corresponding values, we get

$$= \frac{1}{4}\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} + \frac{1}{2}\right)$$

$$= \frac{1}{4}\left(\frac{2-\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}+1+2}{4}\right)$$

$$= \frac{1}{4}\left(\frac{3-\sqrt{5}}{4}\right)\left(\frac{3+\sqrt{5}}{4}\right)$$

$$= \frac{1}{4}\left(\frac{3^2-(\sqrt{5})^2}{4\times 4}\right)$$

$$= \frac{1}{4}\left(\frac{9-5}{16}\right)$$

$$= \frac{1}{16} = \text{RHS}$$

Hence proved

8. Question

Prove that:

$$\cos 36° \cos 42° \cos 60° \cos 78° = \frac{1}{16}$$

Answer

$\text{LHS} = \cos 36° \cos 42° \cos 60° \cos 78°$

By regrouping the LHS and multiplying and dividing by 2 we get,

$$= \frac{1}{2}\cos 36° \cos 60° \left(2\cos 78° \cos 42°\right)$$

But 2cos A cos B = cos (A+B) +cos (A–B)

Then the above equation becomes,

$$= \frac{1}{2}\cos 36° \cos 60° \left(\cos(78° + 42°) + \cos(78° - 42°)\right)$$

$$= \frac{1}{2}\cos 36° \cos 60° \left(\cos(120°) + \cos(36°)\right)$$

$$= \frac{1}{2}\cos 36° \cos 60° \left(\cos(180° - 60°) + \cos(36°)\right)$$

But cos(90°–θ)=sin θ and cos(180°–θ)=–cos(θ).

Then the above equation becomes,

$$= \frac{1}{2}\cos 36° \cos 60° \left(-\cos(60°) + \cos(36°)\right)$$

Now, $\cos(36°) = \frac{\sqrt{5}+1}{4}$

$\cos(60°) = \frac{1}{2}$

Substituting the corresponding values, we get

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1}{4}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}+\frac{\sqrt{5}+1}{4}\right)$$

$$= \left(\frac{\sqrt{5}+1}{16}\right)\left(\frac{\sqrt{5}+1-2}{4}\right)$$

$$= \left(\frac{\left(\sqrt{5}\right)^2 - 1^2}{16 \times 4}\right)$$

$$= \left(\frac{5-1}{64}\right)$$

$$= \frac{1}{16} = \text{RHS}$$

Hence proved

9. Question

Prove that:

$$\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$$

Answer

$$\text{LHS} = \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}$$

This can be rewritten as,

$$= \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\left(\pi - \frac{2\pi}{5}\right)\sin\left(\pi - \frac{\pi}{5}\right)$$

But $\sin(\pi - \theta) = \sin\theta$ so the above equation becomes,

$$= \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{\pi}{5}\right)$$

$$= \sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5}$$

This can be rewritten as,

$$= \sin^2\frac{\pi}{5}\sin^2\left(\frac{\pi}{2} - \frac{\pi}{10}\right)$$

But sin (90°–θ)=cos θ

Then the above equation becomes,

$$= \sin^2\frac{\pi}{5}\cos^2\left(\frac{\pi}{10}\right)$$

Now,

$$\because \cos\frac{\pi}{10} = \frac{\sqrt{10+2\sqrt{5}}}{4}, \sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

Hence the above equation becomes,

$= \left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2 \left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2$

$= \left(\frac{10-2\sqrt{5}}{16}\right)\left(\frac{10+2\sqrt{5}}{16}\right)$

$= \left(\frac{(10)^2-(2\sqrt{5})^2}{16\times 16}\right)$

$= \left(\frac{100-20}{16\times 16}\right)$

$= \left(\frac{80}{16\times 16}\right)$

$= \frac{5}{16} = \text{RHS}$

Hence proved

10. Question

Prove that:

$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{128}$

Answer

$\text{LHS} = \cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$

Multiply and divide by $2\sin\frac{\pi}{15}$, we get

$= \frac{\left(2\sin\frac{\pi}{15}\cos\frac{\pi}{15}\right)\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}}{2\sin\frac{\pi}{15}}$

But 2sin A cos A = sin 2A

Then the above equation becomes,

$= \frac{\left(\sin\frac{2\pi}{15}\right)\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}}{2\sin\frac{\pi}{15}}$

Multiply and divide by 2, we get

$= \frac{\left(2\sin\frac{2\pi}{15}\cos\frac{2\pi}{15}\right)\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}}{2\times 2\sin\frac{\pi}{15}}$

But 2sin A cos A = sin 2A

Then the above equation becomes,

$= \frac{\left(\sin\frac{4\pi}{15}\right)\cos\frac{4\pi}{15}\cos\frac{3\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}}{4\sin\frac{\pi}{15}}$

Multiply and divide by 2, we get

$$= \frac{\left(2\sin\frac{4\pi}{15}\cos\frac{4\pi}{15}\right)\cos\frac{3\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}}{2\times 4\sin\frac{\pi}{15}}$$

But 2sin A cos A = sin 2A

Then the above equation becomes,

$$= \frac{\left(\sin\frac{8\pi}{15}\right)\cos\frac{3\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}}{8\sin\frac{\pi}{15}}$$

Multiply and divide by 2, we get

$$= \frac{\left(2\sin\frac{8\pi}{15}\cos\frac{7\pi}{15}\right)\cos\frac{3\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}}{2\times 8\sin\frac{\pi}{15}}$$

But 2sin A cos B = sin (A+B) +sin(A–B), so the above equation becomes,

$$= \frac{\left(\sin\left(\frac{8\pi}{15}+\frac{7\pi}{15}\right)+\sin\left(\frac{8\pi}{15}-\frac{7\pi}{15}\right)\right)\left(\cos\frac{3\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\right)}{16\sin\frac{\pi}{15}}$$

$$= \frac{\left(\sin(\pi)+\sin\left(\frac{\pi}{15}\right)\right)\left(\cos\frac{3\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\right)}{16\sin\frac{\pi}{15}}$$

$$= \frac{\left(0+\sin\left(\frac{\pi}{15}\right)\right)\left(\cos\frac{3\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\right)}{16\sin\frac{\pi}{15}}$$

$$= \frac{\cancel{\sin\left(\frac{\pi}{15}\right)}\left(\cos\frac{3\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\right)}{16\cancel{\sin\frac{\pi}{15}}}$$

Multiply and divide by $2\sin\frac{3\pi}{15}$, we get

$$= \frac{\left(2\sin\frac{3\pi}{15}\cos\frac{3\pi}{15}\right)\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}}{16\times 2\sin\frac{3\pi}{15}}$$

But 2sin A cos A = sin 2A

Then the above equation becomes,

$$= \frac{\left(\sin\frac{6\pi}{15}\right)\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}}{32\sin\frac{3\pi}{15}}$$

Multiply and divide by 2, we get

$$= \frac{\left(2\sin\frac{6\pi}{15}\cos\frac{6\pi}{15}\right)\cos\frac{5\pi}{15}}{2\times 32\sin\frac{3\pi}{15}}$$

But 2sin A cos A = sin 2A

Then the above equation becomes,

$= \frac{\left(\sin\frac{12\pi}{15}\right)\cos\frac{5\pi}{15}}{64\sin\frac{3\pi}{15}}$

$= \frac{\left(\sin\left(\pi-\frac{3\pi}{15}\right)\right)\left(\cos\frac{5\pi}{15}\right)}{64\sin\frac{3\pi}{15}}$

$= \frac{\left(\sin\left(\frac{3\pi}{15}\right)\right)\left(\cos\frac{5\pi}{15}\right)}{64\sin\frac{3\pi}{15}}$ $(\because \sin(\pi-\theta)=\sin\theta)$

$= \frac{\cos\frac{\pi}{3}}{64}$

$= \frac{\frac{1}{2}}{64}$

$= \frac{1}{128} = \text{RHS}$

Hence proved

Very Short Answer

1. Question

If $\cos 4x = 1 + k\sin^2 x\cos^2 x,$ then write the value of k.

Answer

Given equation is

$\cos 4x = 1 + k \sin^2x \cos^2x$

Now consider the LHS of the equation,

$\cos 4x = 2\cos^2 2x - 1$

[Formula for $\cos 2x = 2\cos^2 x - 1$]

$= 2[2\cos^2 x - 1]^2 - 1$

$= 2[(2\cos^2 x)^2 - 2 \times (2\cos^2 x) \times (1) + (1)^2] - 1$

[Applying $(a-b)^2 = a^2 - 2ab + b^2$ formula]

$= 2[4\cos^4 x - 4\cos^2 x + 1] - 1$

$= 8\cos^4 x - 8\cos^2 x + 2 - 1$

$= 8\cos^2 x (\cos^2 x - 1) + 1$

$= 8\cos^2 x (-\sin^2 x) + 1$

$= -8\cos^2 x \sin^2 x + 1$

Now as per the LHS $\cos 4x = -8\cos^2 x \sin^2 x + 1$ -------- (1)

Comparing LHS with the RHS,

$\cos 4x = 1 - 8\cos^2 x \sin^2 x = 1 + k \sin^2x \cos^2x$

by comparing we get k = -8

2. Question

If $\tan\frac{x}{2}=\frac{m}{n}$, then write the value of m sin x + n cos x.

Answer

Given,

$\tan\frac{x}{y}=\frac{m}{n}$

We need to find the value of m sin x + n cos x

Now consider

m sinx + n cos x $= m\left[\frac{2\tan\frac{x}{y}}{1+\tan^2\frac{x}{y}}\right]+n\left[\frac{1-\tan^2\frac{x}{y}}{1+\tan^2\frac{x}{y}}\right]$

[using the formulas sin 2x & cos 2x in terms of tan x

$\sin 2x=\frac{2\tan x}{1+\tan^2 x}$ and $\cos 2x=\frac{1-\tan^2 x}{1+\tan^2 x}$]

$$= m\left[\frac{2\left(\frac{m}{n}\right)}{1+\left(\frac{m}{n}\right)^2}\right]+n\left[\frac{1-\left(\frac{m}{n}\right)^2}{1+\left(\frac{m}{n}\right)^2}\right]$$

[Substituting tan $\frac{x}{y}=\frac{m}{n}$]

$$= m\left[\frac{2\left(\frac{m}{n}\right)}{\frac{n^2+m^2}{n^2}}\right]+n\left[\frac{\frac{n^2-m^2}{n^2}}{\frac{n^2+m^2}{n^2}}\right]$$

$$= m\left[\frac{2mn}{n^2+m^2}\right]+n\left[\frac{n^2-m^2}{n^2+m^2}\right]$$

$$=\left[\frac{2m^2n}{n^2+m^2}\right]+\left[\frac{n^3-m^2n}{n^2+m^2}\right]$$

$$=\left[\frac{2m^2n+n^3-m^2n}{n^2+m^2}\right]$$

$$=\left[\frac{m^2n+n^3}{n^2+m^2}\right]$$

$$=\left[\frac{n(m^2+n^2)}{m^2+n^2}\right]$$

= n

Hence the value of m sin x + n cos x = n.

3. Question

If $\frac{\pi}{2}<x<\frac{3\pi}{2}$, then write the value of $\sqrt{\frac{1+\cos 2x}{2}}$.

Answer

Given $\frac{\pi}{2}<x<\frac{3\pi}{2}$ then the value of

$$\sqrt{\frac{1+\cos 2x}{2}} = \sqrt{\frac{1+(\cos^2 x - \sin^2 x)}{2}}$$

$$= \sqrt{\frac{\cos^2 x + (1-\sin^2 x)}{2}}$$

$$= \sqrt{\frac{\cos^2 x + \cos^2 x}{2}}$$

$$= \sqrt{\frac{2\cos^2 x}{2}}$$

$$= \sqrt{\cos^2 x}$$

$$= \pm \cos x$$

Hence

$$\sqrt{\frac{1+\cos 2x}{2}} = \pm \cos x$$

But as given, $\frac{\pi}{2} < x < \frac{3\pi}{2}$

This states that, 90° < x < 270°, which means x lies between 2^{nd} and 3^{rd} quadrants.

In the 2^{nd} and 3^{rd} quadrants, the cosine function is negative, so the value of

$$\sqrt{\frac{1+\cos 2x}{2}} = -\cos x$$

4. Question

If $\frac{\pi}{2} < x < \pi$, then write the value of $\sqrt{2+\sqrt{2+2\cos 2x}}$ in the simplest form.

Answer

Given, $\frac{\pi}{2} < x < \Pi$

To find the value of $\sqrt{2+\sqrt{2+2\cos 2x}}$

$$= \sqrt{2+\sqrt{2(1+\cos 2x)}}$$

[using the formula cos 2x = $2\cos^2$ x - 1]

$$= \sqrt{2+\sqrt{2(1+1-2\cos^2 x - 1)}}$$

$$= \sqrt{2+\sqrt{2(2\cos^2 x)}}$$

$$= \sqrt{2+\sqrt{4\cos^2 x}}$$

[using the formula cos 2x = $2\cos^2$ x –1, here 2x = θ so x = $\frac{\theta}{2}$]

$$= \sqrt{2+2\cos x}$$

$= \sqrt{2 + 2\left[2\cos^2\left(\frac{x}{2}\right) - 1\right]}$

$= \sqrt{2 + 4\cos^2\left(\frac{x}{2}\right) - 2}$

$= \sqrt{4\cos^2\left(\frac{x}{2}\right)}$

$= \pm 2\cos\left(\frac{x}{2}\right)$

As given, $\frac{\pi}{2} < x < \Pi$ now by dividing the whole inequation with 2 we get, $\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$.

This clearly state that $\frac{x}{2}$ lies in the 1st quadrant and between 45° and 90°.

So $\sqrt{2 + \sqrt{2 + 2\cos 2x}} = 2\cos\left(\frac{x}{2}\right)$

5. Question

If $\frac{\pi}{2} < x < \pi$, then write the value of $\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$.

Answer

Given, for $\frac{\pi}{2} < x < \pi$ the value of $\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$

Consider,

$\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \sqrt{\frac{1-(\cos^2 x - \sin^2 x)}{1+(\cos^2 x - \sin^2 x)}}$

[by using the formula cos 2x = $\cos^2$ x – $\sin^2$ x]

$= \sqrt{\frac{(1-\cos^2 x) + \sin^2 x}{(1-\sin^2 x) + \cos^2 x}}$

$= \sqrt{\frac{\sin^2 x + \sin^2 x}{\cos^2 x + \cos^2 x}}$

[by using the formula $\cos^2$ x + $\sin^2$ x = 1]

$= \sqrt{\frac{2\sin^2 x}{2\cos^2 x}}$

$= \sqrt{\tan^2 x}$

$= \pm\tan x$

As already mentioned in the question, $\frac{\pi}{2} < x < \pi$, x is in the 2nd quadrant, where tangent function is negative.

Therefore, $\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = -\tan x$

6. Question

If $\pi < x < \frac{2\pi}{2}$, then write the value of $\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$.

Answer

Given, for $\pi < x < \frac{3\pi}{2}$ the value of $\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$

Consider,

$$\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \sqrt{\frac{1-(\cos^2 x - \sin^2 x)}{1+(\cos^2 x - \sin^2 x)}}$$

[by using the formula $\cos 2x = \cos^2 x - \sin^2 x$]

$$= \sqrt{\frac{(1-\cos^2 x) + \sin^2 x}{(1-\sin^2 x) + \cos^2 x}}$$

$$= \sqrt{\frac{\sin^2 x + \sin^2 x}{\cos^2 x + \cos^2 x}}$$

[by using the formula $\cos^2 x + \sin^2 x = 1$]

$$= \sqrt{\frac{2\sin^2 x}{2\cos^2 x}}$$

$$= \sqrt{\tan^2 x}$$

$$= \pm \tan x$$

As already mentioned in the question, $\pi < x < \frac{3\pi}{2}$, x is in the 3^{rd} quadrant, where tangent function is positive.

Therefore, $\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \tan x$

7. Question

In a right-angled triangle ABC, write the value of $\sin^2 A + \sin^2 B + \sin^2 C$.

Answer

Given, triangle ABC is right angle.

So, let ∠ B = 90°

Then as per the property of angles in a triangle

∠ A + ∠ B + ∠ C = 180°

As ∠ B = 90°

∠ A + 90° + ∠ C = 180°

Then ∠ A + ∠ C = 180° - 90° = 90°

Now, consider $\sin^2 A + \sin^2 B + \sin^2 C$

As ∠ B = 90°

$\sin^2 A + \sin^2 B + \sin^2 C = \sin^2 A + \sin^2(90°) + \sin^2 C$

$= \sin^2 A + 1 + \sin^2 C$

From before, we know that ∠ A + ∠ C = 90° ; ∠ C = 90° - ∠ A

$\sin^2 A + \sin^2 B + \sin^2 C = \sin^2 A + 1 + \sin^2(90° - A)$

$= \sin^2 A + \cos^2(A) + 1$

[by using the identity cos x = sin (90° - x)]

$\sin^2 A + \sin^2 B + \sin^2 C = (\sin^2 A + \cos^2 A) + 1$

= 1 + 1

= 2

[by using the identity $\sin^2\theta + \cos^2\theta = 1$]

Therefore, $\sin^2 A + \sin^2 B + \sin^2 C = 2$.

8. Question

Write the value of $\cos^2 76^o + \cos^2 16^o - \cos 76^o \cos 16^o$.

Answer

Given to find the value for,

$\cos^2 76^o + \cos^2 16^o - \cos 76^o \cos 16^o$

In the above expression consider $\cos 76^o \cos 16^o$

[By using the trigonometric sum formula, we can say that,

cos(C+D) + cos (C-D) = 2 cos C cos D]

Now multiply and divide this with 2, we get

$$\frac{2 \times (\cos 76° \cos 16°)}{2} = \frac{\cos(76° + 16°) + \cos(76° - 16°)}{2}$$

$$\frac{\cos 92° + \cos 60°}{2}$$

Consider the full expression,

$$\cos^2 76° + \cos^2 16° - \cos 76° \cos 16°$$

$$= \cos^2 76° + \cos^2 16° - \left(\frac{\cos 92° + \cos 60°}{2}\right)$$

$$= \cos^2 76° + \cos^2 16° - \left(\frac{\cos 92° + \cos 60°}{2}\right)$$

Multiplying and dividing the terms $\cos^2 76° + \cos^2 16°$ with 2

$$= \frac{2\cos^2 76°}{2} + \frac{2\cos^2 16°}{2} - \left(\frac{\cos 92° + \cos 60°}{2}\right)$$

$$= \frac{1}{2}[\cos 2(76) + 1] + \frac{1}{2}[\cos 2(16) + 1] - \left(\frac{\cos 92° + \cos 60°}{2}\right)$$

[by using the formula, $\cos 2\theta = 2\cos^2\theta - 1$ � $2\cos^2\theta = \cos 2\theta + 1$]

$$= \frac{1}{2}[2 + (\cos 152° + \cos 32^o)] - \left(\frac{\cos 92° + \cos 60°}{2}\right)$$

[by using the formula, $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$]

$$= 1 + \frac{1}{2}\left[2\cos\left(\frac{152° + 32°}{2}\right)\cos\left(\frac{152° - 32°}{2}\right)\right] - \left(\frac{\cos 92° + \cos 60°}{2}\right)$$

$= 1 + \frac{1}{2}[2\cos\left(\frac{184°}{2}\right)\cos\left(\frac{120°}{2}\right)] - \left(\frac{\cos 92° + \cos 60°}{2}\right)$

$= 1 + \frac{1}{2}[2\cos(92°)\cos(60°)] - \left(\frac{\cos 92° + \cos 60°}{2}\right)$

$= 1 + \frac{\cos 92°}{2} - \frac{\cos 92°}{2} - \frac{\frac{1}{2}}{2}$

$= 1 - \frac{1}{4} = \frac{3}{4}$

Hence, $\cos^2 76° + \cos^2 16° - \cos 76° \cos 16° = \frac{3}{4}$

9. Question

If $\frac{\pi}{4} < x < \frac{\pi}{2}$, then write the value of $\sqrt{1 - \sin 2x}$.

Answer

Given, $\frac{\pi}{4} < x < \frac{\pi}{2}$

We should find the value for $\sqrt{1 - \sin 2x}$

$\sqrt{1 - \sin 2x} = \sqrt{(\sin^2 x + \cos^2 x) - 2\sin x \cos x}$

[by using the formulae, $\sin^2\theta + \cos^2\theta = 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$]

$\sqrt{1 - \sin 2x} = \sqrt{(\sin^2 x - \cos^2 x)^2}$

$= \sqrt{(\sin x - \cos x)^2}$

$= \pm(\sin x - \cos x)$

As already mentioned in the question, $\frac{\pi}{4} < x < \frac{\pi}{2}$, so x lies in the 1st quadrant and both sine and cosine functions are positive.

Therefore, $\sqrt{1 - \sin 2x} =$ sin x + cos x

10. Question

Write the value of $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$.

Answer

Given expression is $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$

[by using sin2θ = 2 sinθ cosθ � $\cos\theta = \frac{\sin 2\theta}{2\sin\theta}$]

$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = \left(\frac{\sin 2\left(\frac{\pi}{7}\right)}{2\sin\left(\frac{\pi}{7}\right)}\right)\left(\frac{\sin 2\left(\frac{2\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)}\right)\left(\frac{\sin 2\left(\frac{4\pi}{7}\right)}{2\sin\left(\frac{4\pi}{7}\right)}\right)$

$= \left(\frac{\sin 2\left(\frac{\pi}{7}\right)}{2\sin\left(\frac{\pi}{7}\right)}\right)\left(\frac{\sin 2\left(\frac{2\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)}\right)\left(\frac{\sin 2\left(\frac{4\pi}{7}\right)}{2\sin\left(\frac{4\pi}{7}\right)}\right)$

$= \left(\frac{\sin\left(\frac{2\pi}{7}\right)}{2\sin\left(\frac{\pi}{7}\right)}\right)\left(\frac{\sin\left(\frac{4\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)}\right)\left(\frac{\sin\left(\frac{8\pi}{7}\right)}{2\sin\left(\frac{4\pi}{7}\right)}\right)$

$$= \left(\frac{\sin\left(\frac{8\pi}{7}\right)}{2^3 \sin\left(\frac{\pi}{7}\right)}\right) = \left(\frac{\sin\left(\pi + \frac{\pi}{7}\right)}{2^3 \sin\left(\frac{\pi}{7}\right)}\right) = \left(\frac{-\sin\left(\frac{\pi}{7}\right)}{2^3 \sin\left(\frac{\pi}{7}\right)}\right)$$

$$= -\frac{1}{8}$$

Hence $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = -\frac{1}{8}$

11. Question

If $A = \frac{1-\cos B}{\sin B}$, then find the value of tan 2A.

Answer

Given, $\tan A = \frac{1-\cos B}{\sin B}$

To find the value for tan 2A,

Consider

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

[by using the formula for $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$]

$$\tan 2A = \frac{2\left(\frac{1-\cos B}{\sin B}\right)}{1-\left(\frac{1-\cos B}{\sin B}\right)^2}$$

[by substituting the value of tan A as given in the problem]

$$\tan 2A = \frac{2\left(\frac{1-\cos B}{\sin B}\right)}{\frac{\sin^2 B - (1-\cos B)^2}{\sin^2 B}}$$

$$= \frac{2(1-\cos B)\sin B}{\sin^2 B - (1-\cos B)^2}$$

$$= \frac{2(1-\cos B)\sin B}{(1-\cos^2 B) - (1-\cos B)^2}$$

$$= \frac{2(1-\cos B)\sin B}{(1+\cos B)(1-\cos B) - (1-\cos B)^2}$$

$$= \frac{2(1-\cos B)\sin B}{(1-\cos B)[1+\cos B - 1 + \cos B]}$$

$$= \frac{2(1-\cos B)\sin B}{(1-\cos B)2\cos B}$$

$$= \frac{2(1-\cos B)\sin B}{(1-\cos B)2\cos B}$$

$$= \frac{\sin B}{\cos B}$$

= tan B

Therefore, tan 2A = tan B

12. Question

If sin x + cos x = a, find the value of $\sin^6 x + \cos^6 x$.

Answer

Given, sin x + cos x = a

We need to find the value of the expression,

$\sin^6 x + \cos^6 x = (\sin^2 x)^3 + (\cos^2 x)^3$

$= (\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)$

[by using the formula $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$]

$= (1)^3 - 3 \sin^2 x \cos^2 x (1)$

[by using the formula $\sin^2 x + \cos^2 x = 1$]

$$= 1 - 3\left\{\frac{(\sin x + \cos x)^2 - \sin^2 x - \cos^2 x}{2}\right\}^2$$

$$= 1 - 3\left\{\frac{a^2 - (\sin^2 x + \cos^2 x)}{2}\right\}^2$$

[by using the formula $\sin^2 x + \cos^2 x = 1$]

$$= 1 - 3\left\{\frac{a^2 - 1}{2}\right\}^2$$

$$= 1 - \frac{3}{4}(a^2 - 1)^2$$

$$= \frac{4 - 3(a^2 - 1)^2}{4}$$

$$= \frac{1}{4}\{4 - 3(a^2 - 1)^2\}$$

Hence $\sin^6 x + \cos^6 x = \frac{1}{4}\{4 - 3(a^2 - 1)^2\}$

13. Question

If sin x + cos x = a, find the value of |sin x – cos x|

Answer

Given, sin x + cos x = a

To find the value of |sin x – cos x|

Consider square of |sin x – cos x|

$|\sin x - \cos x|^2 = |\sin x|^2 + |\cos x|^2 - 2|\sin x|\,|\cos x|$

[using the formula $(a + b)^2 = a^2 + b^2 + 2ab$]

$|\sin x - \cos x|^2 = |\sin x|^2 + |\cos x|^2 - 2|\sin x|\,|\cos x|$

$= (\sin^2 x + \cos^2 x) - [(\sin x + \cos x)^2 - \sin^2 x - \cos^2 x]$

$= (\sin^2 x + \cos^2 x) - [a^2 - (\sin^2 x + \cos^2 x)]$

[using the formula $\sin^2 x + \cos^2 x = 1$]

$= 1 - a^2 + 1$

$= 2 - a^2$

$|\sin x - \cos x|^2 = 2 - a^2$

Taking square root on both sides.

$\sqrt{|\sin x - \cos x|^2} = \sqrt{2 - a^2}$

Hence $|\sin x - \cos x| = \sqrt{2 - a^2}$

MCQ

1. Question

Mark the Correct alternative in the following:

$8\sin\frac{x}{8}\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}$ is equal to

A.8 cos x

B. cos x

C. 8 sin x

D. sin x

Answer

Given expression, $8\sin\frac{x}{8}\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}$

$4\left(2\sin\frac{x}{8}\cos\frac{x}{8}\right)\cos\frac{x}{2}\cos\frac{x}{4}$

[by rearranging terms]

$4\left(\sin\frac{2x}{8}\right)\cos\frac{x}{2}\cos\frac{x}{4}$

[using the formula $\sin 2\theta = 2\sin\theta\cos\theta$]

$= 4\left(\sin\frac{x}{4}\right)\cos\frac{x}{2}\cos\frac{x}{4}$

$= 2\left(2\sin\frac{x}{4}\cos\frac{x}{4}\right)\cos\frac{x}{2}$

$= 2\left(\sin\frac{2x}{4}\right)\cos\frac{x}{2}$

$= \left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)$

$= \sin x$

Hence $8\sin\frac{x}{8}\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8} = \sin x$

2. Question

Mark the Correct alternative in the following:

$\frac{\sec 8A - 1}{\sec 4A - 1}$ is equal to

A. $\frac{\tan 2A}{\tan 8A}$

B. $\frac{\tan 8A}{\tan 2A}$

C. $\frac{\cot 8A}{\cot 2A}$

D. None of these

Answer

Given expression is $\frac{\sec 8A-1}{\sec 4A-1}$

$$\frac{\sec 8A-1}{\sec 4A-1}=\frac{\frac{1}{\cos 8A}-1}{\frac{1}{\cos 4A}-1}$$

[using $\sec\theta = \frac{1}{\cos\theta}$]

$$=\frac{\frac{1-\cos 8A}{\cos 8A}}{\frac{1-\cos 4A}{\cos 4A}}$$

$$=\frac{\cos 4A\,(1-\cos 8A)}{\cos 8A\,(1-\cos 4A)}$$

$$=\frac{\cos 4A\,\{1-(1-2\sin^2 4A)\}}{\cos 8A\,\{1-(1-2\sin^2 2A)\}}$$

[using $\cos 2\theta = 1 - 2\sin^2\theta$]

$$=\frac{\cos 4A\,(2\sin^2 4A)}{\cos 8A\,(2\sin^2 2A)}$$

$$=\frac{\sin 4A\,(2\sin 4A\cos 4A)}{\cos 8A\,(2\sin^2 2A)}$$

[using $\sin 2\theta = 2\sin\theta\cos\theta$]

$$=\frac{2\sin 2A\cos 2A\,(\sin 8A)}{\cos 8A\,(2\sin^2 2A)}$$

$$=\frac{\cos 2A\,(\sin 8A)}{\cos 8A\,(\sin 2A)}$$

$$=\frac{\left(\frac{\sin 8A}{\cos 8A}\right)}{\left(\frac{\sin 2A}{\cos 2A}\right)}$$

[using $\tan\theta = \frac{\sin\theta}{\cos\theta}$]

$$\frac{\sec 8A-1}{\sec 4A-1}=\frac{\tan 8A}{\tan 2A}$$

3. Question

Mark the Correct alternative in the following:

The value of $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}$ is

A. $\frac{1}{8}$

B. $\frac{1}{16}$

C. $\frac{1}{32}$

D. None of these

Answer

Given expression, $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}$

Multiply and divide the expression with $2\sin\frac{\pi}{65}$

$$=\frac{1}{2\sin\frac{\pi}{65}}\left\{\left(2\sin\frac{\pi}{65}\cos\frac{\pi}{65}\right)\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}\right\}$$

[using the formula sin2θ =2 sin θ cos θ]

$$=\frac{1}{2\sin\frac{\pi}{65}}\left\{\sin\frac{2\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}\right\}$$

Multiply and divide the expression with 2

$$=\frac{1}{2^2\sin\frac{\pi}{65}}\left\{\left(2\sin\frac{2\pi}{65}\cos\frac{2\pi}{65}\right)\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}\right\}$$

[using the formula sin2θ =2 sin θ cos θ]

$$=\frac{1}{2^2\sin\frac{\pi}{65}}\left\{\sin\frac{4\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}\right\}$$

Multiply and divide the expression with 2

$$=\frac{1}{2^3\sin\frac{\pi}{65}}\left\{\left(2\sin\frac{4\pi}{65}\cos\frac{4\pi}{65}\right)\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}\right\}$$

[using the formula sin2θ =2 sin θ cos θ]

$$=\frac{1}{2^3\sin\frac{\pi}{65}}\left\{\sin\frac{8\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}\right\}$$

Multiply and divide the expression with 2

$$=\frac{1}{2^4\sin\frac{\pi}{65}}\left\{\left(2\sin\frac{8\pi}{65}\cos\frac{8\pi}{65}\right)\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}\right\}$$

[using the formula sin2θ =2 sin θ cos θ]

$$=\frac{1}{2^4\sin\frac{\pi}{65}}\left\{\sin\frac{16\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}\right\}$$

Multiply and divide the expression with 2

$$=\frac{1}{2^5\sin\frac{\pi}{65}}\left\{\left(2\sin\frac{16\pi}{65}\cos\frac{16\pi}{65}\right)\cos\frac{32\pi}{65}\right\}$$

[using the formula sin2θ =2 sin θ cos θ]

$$= \frac{1}{2^5 \sin\frac{\pi}{65}}\left\{\sin\frac{32\pi}{65}\cos\frac{32\pi}{65}\right\}$$

Multiply and divide the expression with 2

$$= \frac{1}{2^6 \sin\frac{\pi}{65}}\left\{2\sin\frac{32\pi}{65}\cos\frac{32\pi}{65}\right\}$$

$$= \frac{1}{2^6 \sin\frac{\pi}{65}}\left\{\sin\frac{64\pi}{65}\right\}$$

$$= \frac{1}{2^6 \sin\frac{\pi}{65}}\left\{\sin\left(\pi - \frac{\pi}{65}\right)\right\}$$

$$= \frac{1}{2^6 \sin\frac{\pi}{65}}\left\{\sin\frac{\pi}{65}\right\}$$

$$= \frac{1}{2^6} = \frac{1}{64}$$

As $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65} = \frac{1}{64}$

Hence answer is option D.

4. Question

Mark the Correct alternative in the following:

If cos 2x + 2 cos x = 1 then, (2 – $\cos^2$ x) $\sin^2$ x is equal to

A.1

B. -1

C. $-\sqrt{5}$

D. $\sqrt{5}$

Answer

Given cos 2x + 2 cos x = 1, we need to find the expression,

(2 – $\cos^2$ x) $\sin^2$ x

Consider cos 2x + 2 cos x = 1

2$\cos^2$ x – 1 + 2 cos x -1 = 0

2$\cos^2$ x + 2cos x – 2 = 0

$\cos^2$ x + cos x = 1 -------- (1)

Now consider the expression

(2 – $\cos^2$ x) $\sin^2$ x = (2 – $\cos^2$ x)(1-$\cos^2$x)

= {2 – (1 – cos x)} { 1- (1 – cos x)}

[from equation (1) $\cos^2$ x = 1 - cos x]

= (1+ cos x) (cos x)

= cos x + $\cos^2$ x

[from equation (1) $\cos^2$ x + cos x = 1]

= 1

Hence (2 – $\cos^2$ x) $\sin^2$ x = 1, so option A is the answer.

5. Question

Mark the Correct alternative in the following:

For all real values of x, cot x – 2 cot 2x is equal to

A. tan 2x

B. tan x

C. - cot 3x

D. None of these

Answer

Given expression is cot x – 2 cot 2x for all real values of x

Consider $\cot x - 2\cot 2x = \left(\frac{1}{\tan x}\right) - 2\left(\frac{1-\tan^2 x}{2\tan x}\right)$

[using $\cot x = \left(\frac{1}{\tan x}\right)$ and $\cot 2x = \left(\frac{1-\tan^2 x}{2\tan x}\right)$]

$$= \frac{1 - 1 + \tan^2 x}{\tan x}$$

$$= \frac{\tan^2 x}{\tan x}$$

= tan x

Therefore cot x – 2 cot 2x = tan x.

Option B is the answer.

6. Question

Mark the Correct alternative in the following:

The value of $2\tan\frac{\pi}{10} + 3\sec\frac{\pi}{10} - 4\cos\frac{\pi}{10}$ is

A.0

B. $\sqrt{5}$

C. 1

D. None of these

Answer

Given expression is $2\tan\frac{\pi}{10} + 3\sec\frac{\pi}{10} - 4\cos\frac{\pi}{10}$

Now

$$2\tan\frac{\pi}{10} + 3\sec\frac{\pi}{10} - 4\cos\frac{\pi}{10} = 2\left(\frac{\sin\frac{\pi}{10}}{\cos\frac{\pi}{10}}\right) + 3\left(\frac{1}{\cos\frac{\pi}{10}}\right) - 4\cos\frac{\pi}{10}$$

$$= \frac{2\sin\frac{\pi}{10} + 3 - 4\cos^2\frac{\pi}{10}}{\cos\frac{\pi}{10}}$$

Multiplying and dividing the whole expression with $\cos\frac{\pi}{10}$

$$= \frac{\cos\frac{\pi}{10}\left(2\sin\frac{\pi}{10} + 3 - 4\cos^2\frac{\pi}{10}\right)}{\cos\frac{\pi}{10}\cos\frac{\pi}{10}}$$

$$= \frac{\left(2\sin\frac{\pi}{10}\cos\frac{\pi}{10} + 3\cos\frac{\pi}{10} - 4\cos^3\frac{\pi}{10}\right)}{\cos^2\frac{\pi}{10}}$$

[using sin 2x = 2 sin x cos x formula]

$$= \frac{\sin\frac{2\pi}{10} - \left(4\cos^3\frac{\pi}{10} - 3\cos\frac{\pi}{10}\right)}{\cos^2\frac{\pi}{10}}$$

[using cos 3x = $4\cos^3 x$ – 3 cos x formula]

$$= \frac{\sin\frac{2\pi}{10} - \cos\frac{3\pi}{10}}{\cos^2\frac{\pi}{10}} = \frac{\sin\frac{2\pi}{10} - \sin\left(\frac{\pi}{2} - \frac{2\pi}{10}\right)}{\cos^2\frac{\pi}{10}}$$

$$= \frac{\sin\frac{2\pi}{10} - \sin\left(\frac{\pi}{2} - \frac{3\pi}{10}\right)}{\cos^2\frac{\pi}{10}}$$

[using $\cos x = \sin\left(\frac{\pi}{2} - x\right)$]

$$= \frac{\sin\frac{2\pi}{10} - \sin\left(\frac{2\pi}{10}\right)}{\cos^2\frac{\pi}{10}}$$

= 0

Therefore $2\tan\frac{\pi}{10} + 3\sec\frac{\pi}{10} - 4\cos\frac{\pi}{10} = 0$

The answer is option A.

7. Question

Mark the Correct alternative in the following:

If in a ΔABC, tan A + tan B + tan C = 0, then cot A cot B cot C =-

A.6

B. 1

C. $\frac{1}{6}$

D. None of these

Answer

Given ABC is a triangle, so ∠ A + ∠ B + ∠ C = 180°

Now applying tan on both sides

tan (A+B +C) = tan (180°)

tan (A + B + C) = 0 ----- (1)

Also given tan A + tan B + tan C = 0 ------ (2)

As per the formula of tan (A+B+C)

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

Now, $\tan(A+B+C)=\frac{0\ -\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

[from equation (1)]

$$0=\frac{-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

[from equation (2)]

By cross multiplying

-tan A tan B tan C = 0

tan A tan B tan C = 0

therefore $\frac{1}{\tan A\tan B\ \tan C}=0$

Hence cot A cot B cot C = 0

The answer is option D.

8. Question

Mark the Correct alternative in the following:

If $\cos x=\frac{1}{2}\left(a+\frac{1}{a}\right)$, and $\cos 3x=\lambda\left(a^3+\frac{1}{a^3}\right)$, then λ =

A. $\frac{1}{4}$

B. $\frac{1}{2}$

C. 1

D. None of these

Answer

Given $\cos x=\frac{1}{2}\left(a+\frac{1}{a}\right)$ and $\cos 3x=\lambda\left(a^3+\frac{1}{a^3}\right)$

Consider the equation $\cos 3x=\lambda\left(a^3+\frac{1}{a^3}\right)$

Now take the LHS of the equation,

cos 3x = $4\cos^3 x-3\cos x$

[using the formula for cos 3x = $4\cos^3 x-3\cos x$]

From the question we know, $\cos x=\frac{1}{2}\left(a+\frac{1}{a}\right)$

Substituting the known cos x values in the cos 3x expansion,

$$\cos 3x=4\left[\frac{1}{2}\left(a+\frac{1}{a}\right)\right]^3-3\left[\frac{1}{2}\left(a+\frac{1}{a}\right)\right]$$

$$=4\left[\frac{1}{8}\left(a^3+\frac{1}{a^3}+3a\frac{1}{a}\left(a+\frac{1}{a}\right)\right)\right]-3\left[\frac{1}{2}\left(a+\frac{1}{a}\right)\right]$$

$$= 4\left[\frac{1}{8}\left(a^3+\frac{1}{a^3}\right)+\frac{3}{8}\left(a+\frac{1}{a}\right)\right] - 3\left[\frac{1}{2}\left(a+\frac{1}{a}\right)\right]$$

$$= 4\left[\frac{1}{8}\left(a^3+\frac{1}{a^3}\right)\right] + \frac{3\times 4}{8}\left(a+\frac{1}{a}\right) - 3\left[\frac{1}{2}\left(a+\frac{1}{a}\right)\right]$$

$$= 4\left[\frac{1}{8}\left(a^3+\frac{1}{a^3}\right)\right] + \frac{3}{2}\left(a+\frac{1}{a}\right) - \frac{3}{2}\left(a+\frac{1}{a}\right)$$

$$= 4\left[\frac{1}{8}\left(a^3+\frac{1}{a^3}\right)\right]$$

$\cos 3x = \frac{1}{2}\left(a^3+\frac{1}{a^3}\right)$ ------ (1)

If we compare the RHS of the cos3x equation with the now derived equation (1) we get,

$$\lambda\left(a^3+\frac{1}{a^3}\right) = \frac{1}{2}\left(a^3+\frac{1}{a^3}\right)$$

From the here we can clearly say that $\lambda = \frac{1}{2}$

Hence the answer is option B.

9. Question

Mark the Correct alternative in the following:

If 2 tan α = 3 tan β, then tan (α - β) =

A. $\frac{\sin 2\beta}{5-\cos 2\beta}$

B. $\frac{\cos 2\beta}{5-\cos 2\beta}$

C. $\frac{\sin 2\beta}{5+\cos 2\beta}$

D. None of these

Answer

Given, 2 tan α = 3 tan β

From here we get, $\tan\alpha = \frac{3}{2}\tan\beta$ ------ (1)

Now consider tan (α - β),

The expansion of tan (α - β) is given by

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

As we already know the value of tan α from equation (1), we have,

$$\tan(\alpha-\beta) = \frac{\left(\frac{3}{2}\tan\beta\right) - \tan\beta}{1+\left(\frac{3}{2}\tan\beta\right)\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\left(\frac{3\tan\beta - 2\tan\beta}{2}\right)}{\left(\frac{2+3\tan^2\beta}{2}\right)}$$

$$= \frac{\tan\beta}{2 + 3\tan^2\beta}$$

[by using $\tan\theta = \frac{\sin\theta}{\cos\theta}$]

$$= \frac{\left(\frac{\sin\beta}{\cos\beta}\right)}{2 + 3\left(\frac{\sin\beta}{\cos\beta}\right)^3}$$

$$= \frac{\sin\beta\cos\beta}{2\cos^2\beta + 3\sin^2\beta}$$

$$= \frac{\sin\beta\cos\beta}{2\cos^2\beta + 3(1 - \cos^2\beta)}$$

$$= \frac{\sin\beta\cos\beta}{2\cos^2\beta + 3 - 3\cos^2\beta}$$

$$= \frac{\sin\beta\cos\beta}{3 - \cos^2\beta}$$

Multiplying and dividing the equation with 2

$$= \frac{2\sin\beta\cos\beta}{2(3 - \cos^2\beta)}$$

[using sin2θ = 2 sinθ cosθ]

$$= \frac{\sin 2\beta}{6 - 2\cos^2\beta}$$

In the denominator adding and subtracting 1

$$= \frac{\sin 2\beta}{6 - 2\cos^2\beta + 1 - 1}$$

$$= \frac{\sin 2\beta}{(6 - 1) - (2\cos^2\beta - 1)}$$

[using $\cos 2\theta = 2\cos^2\theta - 1$]

$$\tan(\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}$$

Hence, in the question the answer matches with option A.

10. Question

Mark the Correct alternative in the following:

If $\tan\alpha = \frac{1 - \cos\beta}{\sin\beta}$, then

A.tan 3 α = tan 2 β ok

B. tan 2 α = tan β

C. tan 2 α = tan α

D. None of these

Answer

Given, $\tan A = \frac{1 - \cos B}{\sin B}$

As there are 2 option in terms of tan 2A, let us consider tan 2A

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

[by using the formula for $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$]

$$\tan 2A = \frac{2\left(\frac{1-\cos B}{\sin B}\right)}{1 - \left(\frac{1-\cos B}{\sin B}\right)^2}$$

[by substituting the value of tan A as given in the problem]

$$\tan 2A = \frac{2\left(\frac{1-\cos B}{\sin B}\right)}{\frac{\sin^2 B - (1-\cos B)^2}{\sin^2 B}}$$

$$= \frac{2(1-\cos B)\sin B}{\sin^2 B - (1-\cos B)^2}$$

$$= \frac{2(1-\cos B)\sin B}{(1-\cos^2 B) - (1-\cos B)^2}$$

$$= \frac{2(1-\cos B)\sin B}{(1+\cos B)(1-\cos B) - (1-\cos B)^2}$$

$$= \frac{2(1-\cos B)\sin B}{(1-\cos B)[1+\cos B - 1 + \cos B]}$$

$$= \frac{2(1-\cos B)\sin B}{(1-\cos B)2\cos B}$$

$$= \frac{2(1-\cos B)\sin B}{(1-\cos B)2\cos B}$$

$$= \frac{\sin B}{\cos B}$$

= tan B

Therefore, tan 2A = tan B

Hence the option B is the correct answer.

11. Question

Mark the Correct alternative in the following:

If sin α + sin β = a and cos α – cos β = b, then $\tan\frac{\alpha-\beta}{2} =$

A. $-\frac{a}{b}$

B. $-\frac{b}{a}$

C. $\sqrt{a^2+b^2}$

D. None of these

Answer

Given, sin α + sin β = a and cos α – cos β = b, then the value of

$\tan\frac{\alpha-\beta}{2}$

Consider sin α + sin β = a

As per the expansion of $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

Now , $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = a$ ----- (1)

Similarly, cos α - cos β = b

As per the expansion of $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

Now $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = b$ ------ (2)

By dividing equation (1) with (2) we get,

$$\frac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)} = \frac{a}{b}$$

$$= -\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)} = \frac{a}{b}$$

$$= -\cot\left(\frac{\alpha-\beta}{2}\right) = \frac{a}{b}$$

[As $\tan\theta = \frac{1}{\cot\theta}$]

$$= \tan\left(\frac{\alpha-\beta}{2}\right) = -\frac{b}{a}$$

Therefore the answer is option B.

12. Question

Mark the Correct alternative in the following:

The value of $\left(\cot\frac{x}{2} - \tan\frac{x}{2}\right)^2 (1-2\tan x \cot 2x)$ is

A.1

B. 2

C. 3

D. 4

Answer

Given to find the value of $\left(\cot\frac{x}{2} - \tan\frac{x}{2}\right)^2 (1 - 2\tan x\cot 2x)$

We will solve the expression in two parts,

Now solving 1st term

$$\left(\cot\frac{x}{2} - \tan\frac{x}{2}\right)^2 = \left(\frac{1}{\tan\frac{x}{2}} - \tan\frac{x}{2}\right)^2$$

$= \left(\frac{1}{\tan\frac{x}{2}} - \tan\frac{x}{2}\right)^2$

$= \left(\frac{1 - \tan^2\frac{x}{2}}{\tan\frac{x}{2}}\right)^2$

If we multiply and divide the term by 2, we get,

$= \left(\frac{2\left(1 - \tan^2\frac{x}{2}\right)}{2\tan\frac{x}{2}}\right)^2$

$= 2^2\left(\frac{1 - \tan^2\frac{x}{2}}{2\tan\frac{x}{2}}\right)^2$

[using the formula for $\cot 2x = \frac{1-\tan^2 x}{2\tan x}$ and $\cot x = \frac{1}{\tan x}$]

$= 2^2\left(\frac{1}{\tan x}\right)^2$

$\left(\cot\frac{x}{2} - \tan\frac{x}{2}\right)^2 = \frac{4}{\tan^2 x}$ ----- (1)

Solving the 2nd term

$(1 - 2\tan x \cot 2x) = 1 - 2\tan x\left(\frac{1-\tan^2 x}{2\tan x}\right)$

[using the formula for $\cot 2x = \frac{1-\tan^2 x}{2\tan x}$]

1 – 2 tan x cot 2x = 1 – (1 – $\tan^2$x)

= 1 – 1 + $\tan^2$x

1 – 2 tan x cot 2x = $\tan^2$ x ----- (2)

Now by combining (1) and (2) we get,

$\left(\cot\frac{x}{2} - \tan\frac{x}{2}\right)^2 (1 - 2\tan x \cot 2x) = \left(\frac{4}{\tan^2 x}\right)(\tan^2 x)$

$\left(\cot\frac{x}{2} - \tan\frac{x}{2}\right)^2 (1 - 2\tan x \cot 2x) = 4$

Hence the answer is option D.

13. Question

Mark the Correct alternative in the following:

The value of $\tan x \sin\left(\frac{\pi}{2} + x\right)\cos\left(\frac{\pi}{2} - x\right)$ is

A.1

B. -1

C. $\frac{1}{2}\sin 2x$

D. None of these

Answer

Given to find the value of the expression $\tan x \sin\left(\frac{\pi}{2}+x\right)\cos\left(\frac{\pi}{2}-x\right)$

$\sin\left(\frac{\pi}{2}+x\right)=\sin x$ (as sine is positive in 2nd quadrant)

$\cos\left(\frac{\pi}{2}-x\right)=\sin x$ (as cosine is positive in 1st quadrant)

$\tan x \sin\left(\frac{\pi}{2}+x\right)\cos\left(\frac{\pi}{2}-x\right)=\tan x \cos x \sin x$

$=\frac{\sin x}{\cos x}\cos x \sin x$

$=\sin^2 x$

There for $\tan x \sin\left(\frac{\pi}{2}+x\right)\cos\left(\frac{\pi}{2}-x\right)=\sin^2 x$

Hence the answer is option D.

14. Question

Mark the Correct alternative in the following:

The value of $\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{9}\right)+\sin^2\left(\frac{7\pi}{18}\right)+\sin^2\left(\frac{4\pi}{9}\right)$ is

A.1

B. 2

C. 4

D. None of these

Answer

Given to find the value of $\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{9}\right)+\sin^2\left(\frac{7\pi}{18}\right)+\sin^2\left(\frac{4\pi}{9}\right)$

The angles can be modified as $\frac{7\pi}{18}=\frac{\pi}{2}-\frac{\pi}{9}$ and $\frac{4\pi}{9}==\frac{\pi}{2}-\frac{\pi}{18}$

$\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{9}\right)+\sin^2\left(\frac{7\pi}{18}\right)+\sin^2\left(\frac{4\pi}{9}\right)$

$=\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{9}\right)+\sin^2\left(\frac{\pi}{2}-\frac{\pi}{9}\right)+\sin^2\left(\frac{\pi}{2}-\frac{\pi}{18}\right)$

Using the identity $\sin(\theta)=\cos(\frac{\pi}{2}-\theta)$, we have

$=\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{9}\right)+\cos^2\left(\frac{\pi}{9}\right)+\cos^2\left(\frac{\pi}{18}\right)$

$=\left[\sin^2\left(\frac{\pi}{18}\right)+\cos^2\left(\frac{\pi}{9}\right)\right]+\left[\sin^2\left(\frac{\pi}{9}\right)+\cos^2\left(\frac{\pi}{18}\right)\right]$

[using the identity $\cos^2\theta+\sin^2\theta=1$]

= 1 + 1 = 2

$\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{9}\right)+\sin^2\left(\frac{7\pi}{18}\right)+\sin^2\left(\frac{4\pi}{9}\right)=2$

Hence the answer is option B.

15. Question

Mark the Correct alternative in the following:

If 5 sin α = 3 sin (α + 2 β) ≠ 0, then tan (α + β) is equal to

A.2 tan β

B. 3 tan β

C. 4 tan β

D. 6 tan β

Answer

Given 5 sin α = 3 sin (α + 2 β) ≠ 0, then the value of tan (α + β) is

Consider the given equation,

5 sin α = 3 sin (α + 2 β)

$$\frac{\sin(\alpha + 2\,\beta)}{\sin\alpha} = \frac{5}{3}$$

By applying componendo and dividendo $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$

We get

$$\frac{\sin(\alpha + 2\,\beta) + \sin\alpha}{\sin(\alpha + 2\,\beta) - \sin\alpha} = \frac{5+3}{5-3}$$

[using $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ and $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$ sum of angles]

$$\frac{2\sin\left(\frac{\alpha + 2\beta + \alpha}{2}\right)\cos\left(\frac{\alpha + 2\beta - \alpha}{2}\right)}{2\cos\left(\frac{\alpha + 2\beta + \alpha}{2}\right)\sin\left(\frac{\alpha + 2\beta - \alpha}{2}\right)} = \frac{8}{2}$$

$$\frac{2\sin\left(\frac{2(\alpha + \beta)}{2}\right)\cos\left(\frac{2\beta}{2}\right)}{2\cos\left(\frac{2(\alpha + \beta)}{2}\right)\sin\left(\frac{2\beta}{2}\right)} = 4$$

$$\frac{2\sin(\alpha + \beta)\cos(\beta)}{2\cos(\alpha + \beta)\sin(\beta)} = 4$$

$$\frac{\left[\frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}\right]}{\left[\frac{\sin\beta}{\cos\beta}\right]} = 4$$

$$\frac{\tan(\alpha + \beta)}{\tan\beta} = 4$$

This clearly shows, tan (α + β) = 4 tan β

Hence the answer is option C.

16. Question

Mark the Correct alternative in the following:

The value of 2 cos x – cos 3x – cos 5x – 16 $\cos^3$ x $\sin^2$ x is

A.2

B. 1

C. 0

D. -1

Answer

Given expression is 2 cos x – cos 3x – cos 5x – 16 $\cos^3$ x $\sin^2$ x

Consider the expression

2 cos x – cos 3x – cos 5x – 16 $\cos^3$ x $\sin^2$

= 2 cos x – (cos 5x + cos 3x) – 16 $\cos^3$ x $\sin^2$ x

[using the sum of angles $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$]

$= 2\cos x - \left[2\cos\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)\right] - 16\cos^3 x \sin^2 x$

= 2 cos x – [2 cos 4x cos x] - 16$\cos^3$ x $\sin^2$ x

= 2 cos x (1 – cos 4x) - 16$\cos^3$ x $\sin^2$ x

[using the property cos 2θ = 1- 2 $\sin^2$ θ]

= 2 cos x [1 – (1 – 2 $\sin^2$ 2x)] -16$\cos^3$ x $\sin^2$ x

= 2 cos x [2 $\sin^2$ 2x] -16$\cos^3$ x $\sin^2$ x

= 4cos x [2sin x cos x]^2 -16$\cos^3$ x $\sin^2$ x

[using sin 2θ = 2 sin θ cos θ]

= 4 × 4 (cos x $\sin^2$ x $\cos^2$ x) -16$\cos^3$ x $\sin^2$ x

= 16$\cos^3$ x $\sin^2$ x -16$\cos^3$ x $\sin^2$ x

= 0

Hence cos x – cos 3x – cos 5x – 16 $\cos^3$ x $\sin^2$ x = 0

The answer is option C.

17. Question

Mark the Correct alternative in the following:

If A = 2 $\sin^2$ x – cos 2x, then A lies in the interval

A.[-1, 3]

B. [1, 2]

C. [-2, 4]

D. None of these

Answer

Given A = 2 $\sin^2$ x – cos 2x

[using cos 2x = 1 – 2 $\sin^2$ x]

so A = 2 $\sin^2$ x – cos 2x = 2 $\sin^2$ x –[1 – 2 $\sin^2$ x]

= 2 $\sin^2$ x -1 + 2 $\sin^2$ x]

= 4 $\sin^2$ x – 1

Now A = 2 $\sin^2$ x – cos 2x = 4 $\sin^2$ x – 1

As we know sin x lies between -1 and 1

$-1 \le \sin x \le 1$

$0 \le \sin^2 x \le 1$

Multiplying the inequality by 4

$0 \le 4\sin^2 x \le 4$

Subtracting 1 from the inequality

$-1 \le (4\sin^2 x - 1) \le 3$

From the above inequation, we can say that

A = $(4\sin^2 x - 1)$ belongs to the closed interval [-1,3]

Hence the answer is A.

18. Question

Mark the Correct alternative in the following:

The value of $\frac{\cos 3x}{2\cos 2x - 1}$ is equal to

A.cos x

B. sin x

C. tan x

D. None of these

Answer

Given expression is $\frac{\cos 3x}{2\cos 2x - 1}$

Consider

$$\frac{\cos 3x}{2\cos 2x - 1} = \frac{4\cos^3 x - 3\cos x}{2[2\cos^2 x - 1] - 1}$$

[using the formulae $\cos 3x = 4\cos^3 x - 3\cos x$ and

$\cos 2x = 2\cos^2 x - 1$]

$$\frac{\cos 3x}{2\cos 2x - 1} = \frac{\cos x(4\cos^2 x - 3)}{4\cos^2 x - 2 - 1}$$

$$= \frac{\cos x(4\cos^2 x - 3)}{4\cos^2 x - 3}$$

$= \cos x$

Therefore $\frac{\cos 3x}{2\cos 2x - 1} = \cos x\,\pi$

Hence the answer is option A.

19. Question

Mark the Correct alternative in the following:

If tan (/4 + x) + tan (π/4 – x) = λ sec 2x, then

A.3

B. 4

C. 1

D. 2

Answer

Given equation is

$$\tan\left(\frac{\pi}{4}+x\right) + \tan(\frac{\pi}{4}-x) = \lambda \sec 2x$$

Let us consider LHS

$$\tan\left(\frac{\pi}{4}+x\right) + \tan(\frac{\pi}{4}-x) = \left(\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}\right)+\left(\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)$$

[using the formulae $\tan(A+B)\frac{\tan A+\tan B}{1-\tan A\tan B}$ and $\tan(A-B)\frac{\tan A-\tan B}{1+\tan A\tan B}$]

$$=\left(\frac{1+\tan x}{1-\tan x}\right)+\left(\frac{1-\tan x}{1+\tan x}\right)$$

[the value of tan 45° = 1]

$$=\frac{(1+\tan x)^2+(1-\tan x)^2}{(1+\tan x)(1-\tan x)}$$

$$=\frac{(1+\tan^2 x+2\tan x)+(1+\tan^2 x-2\tan x)}{(1+\tan x)(1-\tan x)}$$

$$=\frac{2(1+\tan^2 x)}{(1-\tan^2 x)}$$

$$=\frac{2\left(1+\frac{\sin^2 x}{\cos^2 x}\right)}{(1-\frac{\sin^2 x}{\cos^2 x})}$$

$$=\frac{2\left(\frac{\cos^2 x+\sin^2 x}{\cos^2 x}\right)}{\left(\frac{\cos^2 x-\sin^2 x}{\cos^2 x}\right)}$$

[using the formulae cos 2x = $\cos^2$ x – $\sin^2$ x and $\cos^2$ x + $\sin^2$ x = 1]

$$=\frac{2(1)}{(\cos 2x)}$$

= 2 sec 2x

Now comparing with the LHS with RHS

$$\tan\left(\frac{\pi}{4}+x\right) + \tan(\frac{\pi}{4}-x) = 2\sec 2x = \lambda \sec 2x$$

From here we can clearly say that the answer is option D.

20. Question

Mark the Correct alternative in the following:

The value of $\cos^2\left(\frac{\pi}{6}+x\right)-\sin^2\left(\frac{\pi}{6}-x\right)$ is

A. $\frac{1}{2}\cos 2x$

B. 0

C. $-\frac{1}{2}\cos 2x$

D. $\frac{1}{2}$

Answer

Given expression is $\cos^2\left(\frac{\pi}{6}+x\right)-\sin^2\left(\frac{\pi}{6}-x\right)$

[using the identity $\sin^2 x + \cos^2 x = 1$]

$$\cos^2\left(\frac{\pi}{6}+x\right)-\sin^2\left(\frac{\pi}{6}-x\right)=1-\sin^2\left(\frac{\pi}{6}+x\right)-\sin^2\left(\frac{\pi}{6}-x\right)$$

$$=1-\left[\sin^2\left(\frac{\pi}{6}+x\right)+\sin^2\left(\frac{\pi}{6}-x\right)\right]$$

[using the formula $a^2 + b^2 = (a + b)^2 - 2ab$]

$$=1-\left[\left(\sin\left(\frac{\pi}{6}+x\right)+\sin\left(\frac{\pi}{6}-x\right)\right)^2-2\sin\left(\frac{\pi}{6}+x\right)\sin\left(\frac{\pi}{6}-x\right)\right]$$

[using the sum of angle formula $\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$]

$$=1-\left[\left(2\sin\left(\frac{\frac{\pi}{6}+x+\frac{\pi}{6}-x}{2}\right)\cos\left(\frac{\frac{\pi}{6}+x-\frac{\pi}{6}+x}{2}\right)\right)^2-2\sin\left(\frac{\pi}{6}+x\right)\sin\left(\frac{\pi}{6}+x\right)\right]$$

$$=1-\left[\left(2\sin\left(\frac{\pi}{6}\right)\cos(x)\right)^2+\left(-2\sin\left(\frac{\pi}{6}+x\right)\sin\left(\frac{\pi}{6}-x\right)\right)\right]$$

[Using the identity cos (A+B) – cos (A-B) = -2sinAsinB]

$$=1-\left[\left(2\left(\frac{1}{2}\right)\cos(x)\right)^2+\left(\cos\left(\frac{\pi}{6}+x+\frac{\pi}{6}-x\right)-\cos\left(\frac{\pi}{6}+x-\frac{\pi}{6}+x\right)\right)\right]$$

$$=1-\left[\cos^2 x+\cos\frac{\pi}{3}-\cos 2x\right]$$

$$=1-\cos^2 x-\frac{1}{2}+\cos 2x$$

[multiplying and dividing the term $\cos^2 x$ with 2]

$$=1-\frac{2\cos^2 x}{2}-\frac{1}{2}+\cos 2x$$

$$=\frac{1}{2}-\frac{2\cos^2 x}{2}+\cos 2x$$

$$=\cos 2x-\left(\frac{2\cos^2 x-1}{2}\right)$$

[using the $\cos 2\theta = 2\cos^2\theta - 1$]

$$=\cos 2x-\frac{1}{2}\cos 2x$$

$$=\frac{1}{2}\cos 2x$$

Hence the answer is option A.

21. Question

Mark the Correct alternative in the following:

$\frac{\sin 3x}{1+2\cos 2x}$ is equal to

A.cos x

B. sin x

C. - cos x

D. sin x

Answer

Given expression $\frac{\sin 3x}{1+2\cos 2x}$

$$\frac{\sin 3x}{1+2\cos 2x} = \frac{3\sin x - 4\sin^3 x}{1+2(1-2\sin^2 x)}$$

[Using the formulae sin3x = 3sinx – 4sin^{3}x and cos2x = 1 – 2sin^{2}x]

$$= \frac{3\sin x - 4\sin^3 x}{1+2-4\sin^2 x}$$

$$= \frac{\sin x(3 - 4\sin^2 x)}{3-4\sin^2 x}$$

= sin x

$$\frac{\sin 3x}{1+2\cos 2x} = \sin x$$

Hence the answer is option B.

22. Question

Mark the Correct alternative in the following:

The value of 2 sin^2 B + 4 cos (A + B) sin A sin B + cos 2 (A + B) is

A.0

B. cos 3 A

C. cos 2A

D. None of these

Answer

Given expression is

2 sin^2 B + 4 cos (A + B) sin A sin B + cos 2 (A + B)

[using the cos (A+B) = cos A cos B – sin A sin B]

= 2 sin^2 B + 4 sin A sin B [cos A cos B - sin A sin B] + cos 2 (A + B)

= 2 sin^2 B + 4 sin A sin B cos A cos B - 4 sin A sin B sin A sin B + cos 2 (A + B)

= 2 sin^2 B + (2 sin A cos A) (2sin B cos B) - 4 sin^{2}A sin^2 B + cos 2 (A + B)

[using sin 2A = 2 sin A cos A]

= 2 $\sin^2 B$ + <u>sin 2A sin 2B</u> - 4 $\sin^2 A \sin^2 B$ + cos (2A + 2B)

=<u>2 $\sin^2 B$ (1 - 2 $\sin^2 A$)</u> + sin 2A sin 2B + <u>(cos2A cos2B - sin 2A sin 2B)</u>

[using cos (A+B) = cos A cos B – sin A sin B]

=2 $\sin^2 B$ (1 - 2 $\sin^2 A$)+ <u>sin 2A sin 2B</u> + cos2A cos2B <u>- sin 2A sin 2B</u>

[using cos 2A = 1 – 2 $\sin^2 x$]

= 2 $\sin^2 B$ cos 2A + cos2A cos2B

= cos 2A (2$\sin^2 B$ + cos 2B)

[using cos 2A = $\cos^2 x - \sin^2 x$]

= cos 2A (2 $\sin^2 B$ + <u>$\cos^2 B - \sin^2 B$</u>)

= cos 2A ($\sin^2 B + \cos^2 B$)

[using the identity $\sin^2 x + \cos^2 x = 1$]

= cos 2A (1)

= cos 2A

Hence

2 $\sin^2 B$ + 4 cos (A + B) sin A sin B + cos 2 (A + B) = cos 2A

The answer is option C.

23. Question

Mark the Correct alternative in the following:

The value of $\frac{2\left(\sin 2x+2\cos^2 x-1\right)}{\cos x-\sin x-\cos 3x+\sin 3x}$ is

A.cos x

B. sec x

C. cosec x

D. sin x

Answer

Given expression is $\frac{2\ (\sin\ 2x+2\cos^2 x-1)}{\cos x-\sin x-\cos 3x+\sin 3x}$

$$\frac{2\ (\sin\ 2x\ +2\ \cos^2 x-1)}{\cos x-\sin x-\cos 3x+\sin 3x}$$

[using cos 2A = $\cos^2 x - \sin^2 x$]

$$=\frac{2\ (\sin\ 2x\ +\cos 2x)}{\cos x-\sin x-\cos 3x+\sin 3x}$$

$$=\frac{2\ (\sin\ 2x\ +\cos 2x)}{(\sin 3x-\sin x)-(\cos 3x-\cos x)}$$

[using $\sin A-\sin B=\ \cos\frac{A+B}{2}\sin\frac{A-B}{2}$ and $\cos A-\cos B=\ -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$]

$$=\frac{2\ (\sin\ 2x\ +\cos 2x)}{\left(2\cos\frac{3x+x}{2}\sin\frac{3x-x}{2}\right)-\left(-2\sin\frac{3x+x}{2}\sin\frac{3x-x}{2}\right)}$$

$= \frac{2\,(\sin\ 2x\ + \cos 2x)}{2\cos 2x \sin x + \ 2\sin 2x \sin x}$

$= \frac{2\ (\sin\ 2x\ + \cos 2x)}{2\sin x\,(\cos 2x + \ \sin 2x)}$

$= \frac{1}{\sin x}$

= cosec x

Therefore $\frac{2\ (\sin\ 2x + 2\cos^2 x - 1)}{\cos x - \sin x - \cos 3x + \sin 3x} = \text{cosec}\ x$

Answer is option C.

24. Question

Mark the Correct alternative in the following:

2(1 – 2 $\sin^2$ 7x) sin 3x is equal to

A.sin 17x – sin 11x

B. sin 11x – sin 17x

C. cos 17x – cos 11x

D. cos 17x + cos 11x

Answer

Given expression is 2(1 – 2 $\sin^2$ 7x) sin 3x

2(1 – 2 $\sin^2$ 7x) sin 3x = 2 cos 2(7x) sin 3x

[using cos 2A = 1 – 2$\sin^2$A]

= 2 cos 14x sin 3x

[using the sum of angles formula $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{1A-B}{2}\right)$]

$= 2\cos\left(\frac{17x + 11x}{2}\right)\sin\left(\frac{17x - 11x}{2}\right)$

= sin (17x) – sin (11x)

Therefore 2(1 – 2 $\sin^2$ 7x) sin 3x = sin (17x) – sin (11x)

The answer is option A.

25. Question

Mark the Correct alternative in the following:

If α and β are acute angles satisfying $\cos 2\alpha = \frac{3\cos 2\beta - 1}{3 - \cos 2\beta}$, then tan α =

A. $\sqrt{2}\ \tan\ \beta$

B. $\frac{1}{\sqrt{2}}\tan\beta$

C. $\sqrt{2}\ \cot\ \beta$

D. $\frac{1}{\sqrt{2}}\cot\beta$

Answer

Given for α < 90° and β < 90° , $\cos 2\alpha = \frac{3\cos 2\beta - 1}{3 - \cos 2\beta}$

Then tan α is given by

Consider

$$\frac{\cos 2\alpha}{1} = \frac{2\cos 2\beta - 1}{3 - \cos 2\beta}$$

[using componendo and dividend principle, if $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$]

$$\frac{\cos 2\alpha + 1}{\cos 2\alpha - 1} = \frac{(3\cos 2\beta - 1) + (3 - \cos 2\beta)}{(3\cos 2\beta - 1) - (3 - \cos 2\beta)}$$

$$\frac{(1 - 2\sin^2\alpha) + 1}{(2\cos^2\alpha - 1) - 1} = \frac{(2\cos 2\beta + 2)}{(4\cos 2\beta - 4)}$$

[using cos 2x = 1- 2sin^2 x = 2cos^{2}x - 1]

$$\frac{2(1 - \sin^2\alpha)}{-2(1 - \cos^2\alpha)} = \frac{2(\cos 2\beta + 1)}{4(\cos 2\beta - 1)}$$

[using cos 2x = cos^2 x - sin^2 x]

$$-\frac{\cos^2\alpha}{\sin^2\alpha} = \frac{(\cos^2\beta - \sin^2\beta + 1)}{2(\cos^2\beta - \sin^2\beta - 1)}$$

$$-\frac{\cos^2\alpha}{\sin^2\alpha} = \frac{(\cos^2\beta + 1 - \sin^2\beta)}{-2(1 - \cos^2\alpha + \sin^2 x)}$$

[using cos^2 x + sin^2 x = 1]

$$-\frac{\cos^2\alpha}{\sin^2\alpha} = -\frac{2(\cos^2\beta)}{4(\sin^2\beta)}$$

$$\frac{1}{\tan^2\alpha} = \frac{1}{2\tan^2\beta}$$

$\tan^2 \alpha = 2\tan^2 \beta$

applying square root on both sides

$$\sqrt{\tan^2\alpha} = \sqrt{2\tan^2\beta}$$

$$\tan\alpha = \sqrt{2}\tan\beta$$

Hence the answer is option A.

26. Question

Mark the Correct alternative in the following:

If $\tan\frac{x}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\alpha}{2}$, then cos α =

A.1 – e cos (cos x + e)

B. $\dfrac{1+e\cos x}{\cos x-e}$

C. $\dfrac{1-e\cos x}{\cos x-e}$

D. $\dfrac{\cos x-e}{1-e\cos x}$

Answer

Given $\tan\frac{x}{2}=\sqrt{\frac{1-e}{1+e}}\tan\frac{\alpha}{2}$, then cos α is

Let

$$\tan\frac{\alpha}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{x}{2}$$

By using the expansion of cos 2x in terms of tan x

$$\cos 2x=\frac{1-\tan^2 x}{1+\tan^2 x}$$

We get,

$$\cos\alpha=\frac{1-\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{x}{2}\right)^2}{1+\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{x}{2}\right)^2}$$

$$=\frac{1-\left(\frac{1+e}{1-e}\tan^2\frac{x}{2}\right)}{1+\left(\frac{1+e}{1-e}\tan^2\frac{x}{2}\right)}$$

$$=\frac{1-e-\left[(1+e)\tan^2\frac{x}{2}\right]}{1-e+\left[(1+e)\tan^2\frac{x}{2}\right]}$$

$$=\frac{1-e-\tan^2\frac{x}{2}-e\tan^2\frac{x}{2}}{1-e+\tan^2\frac{x}{2}+e\tan^2\frac{x}{2}}$$

$$=\frac{1-\tan^2\frac{x}{2}-e-e\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}-e+e\tan^2\frac{x}{2}}$$

$$=\frac{\left(1-\tan^2\frac{x}{2}\right)-e\left(1+\tan^2\frac{x}{2}\right)}{\left(1+\tan^2\frac{x}{2}\right)-e\left(1-\tan^2\frac{x}{2}\right)}$$

Dividing the numerator and denominator by $1+\tan^2\frac{x}{2}$

$$=\frac{\frac{\left(1-\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}-\frac{e\left(1+\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}}{\frac{\left(1+\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}-\frac{e\left(1-\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}}$$

[using the formula for cos 2x in terms of tan x $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$]

$$= \frac{\cos x - e}{1 - e\cos x}$$

Hence the answer is option D.

27. Question

Mark the Correct alternative in the following:

If $(2^n + 1)\, x = \pi$, then $2^n \cos x \cos 2x \cos 2^2 x \ldots\ldots \cos 2^{n-1} x =$

A.-1

B. 1

C. 1/2

D. None of these

Answer

Given $(2^n - 1)\, x = \pi$

Then evaluate the expression

$2^n \cos x \cos 2x \cos 2^2 x \ldots\ldots \cos 2^{n-1} x$

by taking a 2 from 2^n and multiplying and dividing by sin x, we get

$$= \frac{2^{n-1}}{\sin x}(2\sin x \cos x)\cos 2x \cos 2^2 x \ldots\ldots\ldots\ldots \cos 2^{n-1} x$$

[by using the formula sin 2x = 2 sin x cos x]

$$= \frac{2^{n-1}}{\sin x}(\sin 2x)\cos 2x \cos 2^2 x \ldots\ldots\ldots\ldots \cos 2^{n-1} x$$

Now borrowing another 2 from 2^{n-1}

$$= \frac{2^{n-2}}{\sin x}(2\ \sin 2x \cos 2x)\cos 2^2 x \ldots\ldots\ldots\ldots \cos 2^{n-1} x$$

$$= \frac{2^{n-2}}{\sin x}(\sin 4x)\cos 4x \ldots\ldots\ldots\ldots \cos 2^{n-1} x$$

These iterations repeat till we reach the last term

$$= \frac{2^{n-(n-1)}}{\sin x}\sin 2^{n-1} x \cos 2^{n-1} x$$

$$= \frac{2\ \sin 2^{n-1} x \cos 2^{n-1} x}{\sin x}$$

$$= \frac{\sin 2^n x}{\sin x}$$

As already given that

$2^n x + x = 180°$

$2^n x = 180° - x$

So substituting the same in the above solution

$$2^n \cos x \cos 2x \cos 2^2 x \ldots\ldots\ldots\ldots \cos 2^{n-1} x = \frac{\sin(\pi - x)}{\sin x} == \frac{\sin x}{\sin x} = 1$$

So the answer is option B.

28. Question

Mark the Correct alternative in the following:

If tan x = t then tan 2x + sec 2x is equal to

A. $\frac{1+t}{1-t}$

B. $\frac{1-t}{1+t}$

C. $\frac{2t}{1-t}$

D. $\frac{2t}{1+t}$

Answer

Given tan x = t

then tan 2x + sex 2x =

[using the formulae for tan 2x and sec 2x in terms of tan x,

$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$ and $\sec 2x = \frac{1+\tan^2 x}{1-\tan^2 x}$]

Now

$$\tan 2x + \sec 2x = \frac{2\tan x}{1-\tan^2 x} + \frac{1+\tan^2 x}{1-\tan^2 x}$$

$$= \frac{2\tan x + 1 + \tan^2 x}{1-\tan^2 x}$$

$$= \frac{(1+\tan x)^2}{(1+\tan x)(1-\tan x)}$$

$$= \frac{(1+\tan x)}{(1-\tan x)}$$

As already given tan x = t

$$\tan 2x + \sec 2x = \frac{1+t}{1-t}$$

Hence the answer is option A.

29. Question

Mark the Correct alternative in the following:

The value of $\cos^4 x + \sin^4 x - 6\cos^2 x \sin^2 x$ is

A.cos 2x

B. sin 2x

C. cos 4x

D. None of these

Answer

Given expression is $\cos^4 x + \sin^4 x - 6\cos^2 x \sin^2 x$

$=[(\cos^2x)^2 + (\sin^2x)^2 - 2\cos^2x \sin^2x] - 4\cos^2x \sin^2x$

[using the formula $a^2 + b^2 = (a+b)^2 - 2ab$]

$= (\cos^2x - \sin^2x)^2 - 4\cos^2x \sin^2x$

[using the formula $\cos 2x = \cos^2 x - \sin^2 x$]

$= (\cos 2x)^2 - (2 \sin x \cos x)^2$

[using the formula $\sin 2x = 2 \sin x \cos x$]

$= (\cos 2x)^2 - (\sin 2x)^2$

[using the formula $\cos 2x = \cos^2 x - \sin^2 x$]

$= \cos 4x$

Therefore $\cos^4 x + \sin^4 x - 6\cos^2 x \sin^2 x = \cos 4x$

The answer is option A.

30. Question

Mark the Correct alternative in the following:

The value of $\cos(36^o - A)\cos(36^o + A) + \cos(54^o - A)\cos(54^o + A)$ is

A. $\cos 2A$

B. $\sin 2A$

C. $\cos A$

D. 0

Answer

Given expression

$\cos(36^o - A)\cos(36^o + A) + \cos(54^o - A)\cos(54^o + A)$

In the above expression angle $\cos(54° + A) = \sin[90° - (54° + A)]$

And $\cos(54° - A) = \sin[90° - (54° + A)]$

[using $\cos\theta = \sin(90° - \theta)$]

Now substituting the same in the expression

$= \cos(36^o - A)\cos(36^o + A) + \sin[90°-(54°-A)]\sin[90°-(54° + A)]$

$= \cos(36^o - A)\cos(36^o + A) + \sin(36^o + A)\sin(36^o - A)$

$= \cos(36^o + A)\cos(36^o - A) + \sin(36^o + A)\sin(36^o - A)$

[using $\cos(A-B) = \cos A \cos B + \sin A \sin B$]

$= \cos[(36° + A) - (36° - A)]$

$= \cos(2A)$

Therefore the answer is option A.

31. Question

Mark the Correct alternative in the following:

The value of $\tan x \tan\left(\frac{\pi}{3}-x\right)\tan\left(\frac{\pi}{3}+x\right)$ is

A.cot 3x

B. 2 cot 3x

C. tan 3x

D. 3 tan 3x

Answer

Given expression is $\tan x \tan\left(\frac{\pi}{3}-\ x\right)\tan\left(\frac{\pi}{3}+\ x\right)$

[using $\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$ and $\tan(A-\ B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$]

Then

$\tan x \tan\left(\frac{\pi}{3}-\ x\right)\tan\left(\frac{\pi}{3}+\ x\right)$

$= \tan x\left(\frac{\tan\frac{\pi}{3}-\tan x}{1+\tan\frac{\pi}{3}\tan x}\right)\left(\frac{\tan\frac{\pi}{3}+\tan x}{1-\tan\frac{\pi}{3}\tan x}\right)$

$= \tan x\left(\frac{\sqrt{3}-\tan x}{1+\sqrt{3}\tan x}\right)\left(\frac{\sqrt{3}+\tan x}{1-\sqrt{3}\tan x}\right)$

[using $a^2 - b^2 = (a-b)(a+b)$]

$= \tan x\left(\frac{(\sqrt{3})^2-\tan^2 x}{1-\ (\sqrt{3})^2\tan^2 x}\right)$

$= \tan x\left(\frac{3-\tan^2 x}{1-\ 3\tan^2 x}\right)$

$= \left(\frac{3\tan x-\tan^3 x}{1-\ 3\tan^2 x}\right)$

[using $\tan 3x = \left(\frac{3\tan x-\tan^3 x}{1-\ 3\tan^2 x}\right)$ formula]

= tan 3x

Therefore $\tan x \tan\left(\frac{\pi}{3}-\ x\right)\tan\left(\frac{\pi}{3}+\ x\right) = \tan 3x$

The answer is option C.

32. Question

Mark the Correct alternative in the following:

The value $\tan x+\tan\left(\frac{\pi}{3}+\ x\right)+\tan\left(\frac{2\pi}{3}+\ x\right)$ of is

A.3 tan 3x

B. tan 3x

C. 3 cot 3x

D. cot 3x

Answer

Given $\tan x + \tan\left(\frac{\pi}{3} + x\right) + \tan\left(\frac{2\pi}{3} + x\right)$

[using $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$]

Then

$$\tan x + \tan\left(\frac{\pi}{3} + x\right) + \tan\left(\frac{2\pi}{3} + x\right)$$

$$= \tan x + \left(\frac{\tan\frac{\pi}{3} + \tan x}{1 - \tan\frac{\pi}{3}\tan x}\right) + \left(\frac{\tan\frac{2\pi}{3} + \tan x}{1 - \tan\frac{2\pi}{3}\tan x}\right)$$

$$= \tan x + \left(\frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}\right) + \left(\frac{-\sqrt{3} + \tan x}{1 - (-\sqrt{3})\tan x}\right)$$

$$= \tan x + \left(\frac{\tan x + \sqrt{3}}{1 - \sqrt{3}\tan x}\right) + \left(\frac{\tan x - \sqrt{3}}{1 + \sqrt{3}\tan x}\right)$$

$$= \tan x + \frac{(\tan x + \sqrt{3})(1 + \sqrt{3}\tan x) + (\tan x - \sqrt{3})(1 - \sqrt{3}\tan x)}{(1 - \sqrt{3}\tan x)(1 + \sqrt{3}\tan x)}$$

[using $a^2 - b^2 = (a-b)(a+b)$]

$$= \tan x + \frac{(\tan x + \sqrt{3}\tan^2 x + \sqrt{3} + 3\tan x) + (\tan x - \sqrt{3}\tan^2 x - \sqrt{3} + 3\tan x)}{1 - 3\tan^2 x}$$

$$= \frac{\tan x(1 - 3\tan^2 x) + 8\tan x}{1 - 3\tan^2 x}$$

$$= \frac{\tan x - 3\tan^3 x + 8\tan x}{1 - 3\tan^2 x}$$

$$= \frac{9\tan x - 3\tan^3 x}{1 - 3\tan^2 x}$$

$$= \frac{3(3\tan x - \tan^3 x)}{1 - 3\tan^2 x}$$

[using $\tan 3x = \left(\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\right)$ formula]

= 3 tan 3x

Therefore $\tan x + \tan\left(\frac{\pi}{3} + x\right) + \tan\left(\frac{2\pi}{3} + x\right) = 3\tan 3x$

The answer is option A.

33. Question

Mark the Correct alternative in the following:

The value of is $\frac{\sin 5\alpha - \sin 3\alpha}{\cos 5\alpha + 2\cos 4\alpha + \cos 3\alpha}$

A.cot α/2

B. cot α

C. tan α/2

D. None of these

Answer

Given

$$\frac{\sin 5\alpha - \sin 3\alpha}{\cos 5\alpha + 2\cos 4\alpha + \cos 3\alpha}$$

[Using $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$]

$$= \frac{2\cos\left(\frac{5\alpha+3\alpha}{2}\right)\sin\left(\frac{5\alpha-3\alpha}{2}\right)}{2\cos\left(\frac{5\alpha+3\alpha}{2}\right)\cos\left(\frac{5\alpha-3\alpha}{2}\right) + 2\cos 4\alpha}$$

$$= \frac{2\cos 4\alpha \sin\alpha}{2\cos 4\alpha\cos\alpha + 2\cos 4\alpha}$$

$$= \frac{2\cos 4\alpha \sin\alpha}{2\cos 4\alpha(\cos\alpha + 1)}$$

$$= \frac{\sin\alpha}{(\cos\alpha + 1)}$$

[using $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$ and $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$]

$$= \frac{\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}}{\left(\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\right)+1}$$

$$= \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}+1+\tan^2\frac{\alpha}{2}}$$

$$= \frac{2\tan\frac{\alpha}{2}x}{2}$$

$= \tan\frac{\alpha}{2}$

Therefore $\frac{\sin 5\alpha - \sin 3\alpha}{\cos 5\alpha + 2\cos 4\alpha + \cos 3\alpha} = \tan\frac{\alpha}{2}$

Answer is option C.

34. Question

Mark the Correct alternative in the following:

$\frac{\sin 5x}{\sin x}$ is equal to

A.16 $\cos^4 x - 12\cos^2 x + 1$

B. 16 $\cos^4 x + 12\cos^2 x + 1$

C. 16 $\cos^4 x - 12\cos^2 x - 1$

D. 16 $\cos^4 x + 12\cos^2 x - 1$

Answer

Given $\frac{\sin 5x}{\sin x}$

Let 5x = 3x + 2x

Then

$$\frac{\sin 5x}{\sin x} = \frac{\sin(3x+2x)}{\sin x}$$

[using sin (A+B) = sin A cos B + cos A sin B]

$$= \frac{\sin 3x \cos 3x + \cos 3x \sin 2x}{\sin x}$$

[using the formulae :

$\sin 3x = 3\sin x - 4\sin^3 x$

$\cos 3x = 4\cos^3 x - 3\cos x$

$\cos 2x = 2\cos^2 x - 1$

$\sin 2x = 2\sin x\cos x$]

$$= \frac{(3\sin x - 4\sin^3 x)(2\cos^2 x - 1) + (4\cos^3 x - 3\cos x)(2\sin x\cos x)}{\sin x}$$

$$= \frac{\sin x(3 - 4\sin^2 x)(2\cos^2 x - 1) + \sin x(4\cos^3 x - 3\cos x)(2\cos x)}{\sin x}$$

$$= \frac{\sin x[(3 - 4\sin^2 x)(2\cos^2 x - 1) + (4\cos^3 x - 3\cos x)(2\cos x)]}{\sin x}$$

$= (3 - 4\sin^2 x)(2\cos^2 x - 1) + (4\cos^3 x - 3\cos x)(2\cos x)$

$= (6\cos^2 x - 3 - 8\sin^2 x\cos^2 x + 4\sin^2 x) + (8\cos^4 x - 6\cos^2 x)$

[using $\sin^2 x + \cos^2 x = 1$]

$= -3 - 8(1 - \cos^2 x)\cos^2 x + 4(1 - \cos^2 x) + 8\cos^4 x$

$= -3 - 8\cos^2 x + 8\cos^4 x + 4 - 4\cos^2 x + 8\cos^4 x$

$= 16\cos^4 x - 12\cos^2 x + 1$

Therefore the answer is option A.

35. Question

Mark the Correct alternative in the following:

If n = 1, 2, 3, ..., then $\cos\alpha \cos 2\alpha \cos 4\alpha \ldots \cos 2^{n-1}\alpha$ is equal to

A. $\frac{\sin 2n\alpha}{2n\sin\alpha}$

B. $\frac{\sin 2^n\alpha}{2^n \sin 2^{n-1}\alpha}$

C. $\frac{\sin 4^{n-1}\alpha}{4^{n-1}\sin\alpha}$

D. $\frac{\sin 2^n\alpha}{2^n\sin\alpha}$

Answer

Given expression

$\cos \alpha \cos 2\alpha \cos 4\alpha \ldots \cos 2^{n-1}\alpha$

multiplying and dividing the expression by 2 sin α , we get,

$$= \frac{1}{2\sin\alpha}(2\sin\alpha\cos\alpha)\cos 2\alpha\cos 4\alpha \ldots\ldots\ldots\ldots\ldots \cos 2^{n-1}\alpha$$

[using sin 2x = 2 sin x cos x]

$$= \frac{1}{2\sin\alpha}(\sin 2\alpha)\cos 2\alpha\cos 4\alpha \ldots\ldots\ldots\ldots\ldots \cos 2^{n-1}\alpha$$

Now multiplying and dividing the expression with 2.

$$= \frac{1}{2^2\sin\alpha}(2\sin 2\alpha\cos 2\alpha)\cos 4\alpha \ldots\ldots\ldots\ldots\ldots \cos 2^{n-1}\alpha$$

$$= \frac{1}{2^2\sin\alpha}(\sin 4\alpha)\cos 4\alpha \ldots\ldots\ldots\ldots\ldots \cos 2^{n-1}\alpha$$

Continuing this process for n-1 times we will get

$$= \frac{1}{2^{n-1}\sin\alpha}\sin 2^{n-1}\alpha\cos 2^{n-1}\alpha$$

Now repeating for the last time,

$$= \frac{1}{(2^{n-1}\times 2)\sin\alpha}(2\sin 2^{n-1}\alpha\cos 2^{n-1}\alpha)$$

$$= \frac{1}{2^n\sin\alpha}(\sin 2^n\alpha)$$

This proves that

$$\cos\alpha)\cos 2\alpha\cos 4\alpha \ldots\ldots\ldots\ldots\ldots \cos 2^{n-1}\alpha = \frac{\sin 2^n\alpha}{2^n\sin\alpha}$$

Hence the answer is option D.

36. Question

Mark the Correct alternative in the following:

If $\tan x = \frac{a}{b}$, then b cos 2x + a sin 2x is equal to

A.a

B. b

C. $\frac{a}{b}$

D. $\frac{b}{a}$

Answer

Given $\tan x = \frac{a}{b}$

The value of the expression b cos 2x + a sin 2x

Now consider b cos 2x + a sin 2x

[by using $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$ and $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$]

$$b\cos 2x + a\sin 2x = b\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right) + a\left(\frac{2\tan x}{1+\tan^2 x}\right)$$

As already given $\tan x = \frac{a}{b}$

Then

$$b\cos 2x + a\sin 2x = b\left(\frac{1-\left(\frac{a}{b}\right)^2}{1+\left(\frac{a}{b}\right)^2}\right) + a\left(\frac{2\frac{a}{b}}{1+\left(\frac{a}{b}\right)^2}\right)$$

$$= b\left(\frac{\frac{b^2-a^2}{b^2}}{\frac{b^2+a^2}{b^2}}\right) + a\left(\frac{2\frac{a}{b}}{\frac{b^2+a^2}{b^2}}\right)$$

$$= b\left(\frac{b^2-a^2}{b^2+a^2}\right) + a\left(\frac{2ab}{b^2+a^2}\right)$$

$$= \left(\frac{b^3-a^2b}{b^2+a^2}\right) + \left(\frac{2a^2b}{b^2+a^2}\right)$$

$$= \left(\frac{b^3-a^2b+2a^2b}{b^2+a^2}\right)$$

$$= \left(\frac{b^3+a^2b}{b^2+a^2}\right)$$

$$= \frac{b(b^2+a^2)}{b^2+a^2}$$

= b

Hence b cos 2x + a sin 2x = b.

The answer is option B.

37. Question

Mark the Correct alternative in the following:

If $\tan\alpha = \frac{1}{7}, \tan\beta = \frac{1}{3}$, then cos 2α is equal to

A.sin 2β

B. sin 4β

C. sin 3β

D. cos 2β

Answer

Given $\tan\alpha = \frac{1}{7}$ and $\tan\beta = \frac{1}{3}$

Now to find the value of cos 2α

[By using $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$]

$$\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

[as $\tan\alpha = \frac{1}{7}$ is given]

$$\cos 2\alpha = \frac{1-\left(\frac{1}{7}\right)^2}{1+\left(\frac{1}{7}\right)^2}$$

$$= \frac{49-1}{49+1}$$

$$= \frac{48}{50} = \frac{24}{25}$$

Hence $\cos 2\alpha = \frac{24}{25}$

The same value is obtained for sin 4β.

[By sin 2x = 2 sinx cosx]

$$\sin 4\alpha = 2\sin 2\alpha\cos 2\alpha$$

[using $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$ and $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$]

We have

$$\sin 4\beta = 2\left(\frac{2\tan\beta}{1+\tan^2\beta}\right)\left(\frac{1-\tan^2\beta}{1+\tan^2\beta}\right)$$

As $\tan\beta = \frac{1}{3}$

$$\sin 4\beta = 2\left(\frac{2\left(\frac{1}{3}\right)}{1+\left(\frac{1}{3}\right)^2}\right)\left(\frac{1-\left(\frac{1}{3}\right)^2}{1+\left(\frac{1}{3}\right)^2}\right)$$

$$= 2\left(\frac{6}{9+1}\right)\left(\frac{9-1}{9+1}\right)$$

$$= 2\left(\frac{48}{100}\right) = \frac{48}{50} = \frac{24}{25}$$

As the value of cos 2α and sin 4α are the same, the answer is option B.

38. Question

Mark the Correct alternative in the following:

The value of $\cos^2 48^o - \sin^2 12^o$ is

A. $\frac{\sqrt{5}+1}{8}$

B. $\frac{\sqrt{5}-1}{8}$

C. $\frac{\sqrt{5}+1}{5}$ D.

$\frac{\sqrt{5}+1}{2\sqrt{2}}$

Answer

Given

$\cos^2 48^o - \sin^2 12^o$

[by using the formula cos 2x = $2\cos^2 x - 1$ and $\cos 2x = 1 - 2\sin^2 x$]

$$\cos^2 48^o - \sin^2 12^o = \left(\frac{\cos(2 \times 48^o) + 1}{2}\right) - \left(\frac{1 - \cos(2 \times 12^o)}{2}\right)$$

$$= \left(\frac{\cos(96^o) + 1}{2}\right) - \left(\frac{1 - \cos(24^o)}{2}\right)$$

$$= \left(\frac{\cos(96^o) + 1 - 1 + \cos(24^o)}{2}\right)$$

$$= \left(\frac{\cos(96^o) + \cos(24^o)}{2}\right)$$

[by using the formula $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$]

$$= \frac{1}{2}\left[2\cos\left(\frac{96^o + 24^o}{2}\right)\cos\left(\frac{96^o - 24^o}{2}\right)\right]$$

$$= \cos\left(\frac{120^o}{2}\right)\cos\left(\frac{72^o}{2}\right)$$

$$= \cos(60^o)\cos(36^o)$$

$$= \frac{1}{2}\left(\frac{1 + \sqrt{5}}{4}\right)$$

$$= \frac{1 + \sqrt{5}}{8}$$

Therefore $\cos^2 48^o - \sin^2 12^o = \frac{1+\sqrt{5}}{8}$

Hence the answer is option A.